3-7 Applied Calculus Solutions

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1. The demand and supply functions for a certain product are given by $p = 150 - .5q$ and $p = .002q^2 + 1.5$, where p is in dollars and q is the number of items.
   (a) Which is the demand function?
   (b) Find the equilibrium price and quantity
   (c) Find the total gains from trade at the equilibrium price.

(1a) The demand function is the decreasing function.

For the first function, $\frac{dp}{dq} = \frac{d(150 - .5q)}{dq} = -0.5 < 0$

For the second function, $\frac{dp}{dq} = \frac{d(0.002q^2 + 1.5)}{dq} = 0.004q \geq 0$

$\therefore$ the demand function is $\boxed{p = 150 - .5q}$

(1b) The equilibrium price is p^* and the equilbrium quantity is q^*, where (q^*, p^*) is the intersection of the supply curve and the demand curve.

$150 - .5q = .002q^2 + 1.5$

$\Rightarrow .002q^2 + .5q - 150 + 1.5 = 0$

$\Rightarrow .002q^2 + .5q - 148.5 = 0$

$$q = \frac{-.5 \pm \sqrt{(.5)^2 - 4(.002)(-148.5)}}{(2)(.002)} = \frac{-.5 \pm \sqrt{1.438}}{0.004}$$

The quantity cannot be negative, so

$q = \frac{-.5 + \sqrt{1.438}}{0.004} = 174.79 \approx \boxed{175 \text{ items}}$

$\Rightarrow p = 150 - .5(175) = \boxed{\$62.5}$

$\boxed{(q^*, p^*) = (175 \text{ items}, \$62.5)}$

(1c)

Consumer and Producer Surplus

Given a demand function p = d(q) and a supply function p = s(q), and the equilibrium point (q*, p*)

The **consumer surplus** = $\int_0^{q^*} d(q)dq - p^*q^*$

The **producer surplus** = $p^*q^* - \int_0^{q^*} s(q)dq$

The sum of the consumer surplus and producer surplus is the **total gains from trade**.

$$\int_0^{q^*} d(q)dq - p^*q^* = \int_0^{175} (150 - .5q)dq - (62.5)(175)$$

$$= \left[150q - \frac{.5q^2}{2}\right]_0^{175} - 10938.$$

$$= \left(150(175) - \frac{.5(175)^2}{2}\right) - 0 - 10938 = \boxed{7655.8}$$

$$p^*q^* - \int_0^{q^*} s(q)dq = (62.5)(175) - \int_0^{175} (.002q^2 + 1.5)dq$$

$$= 10938. - \left[\frac{.002q^3}{3} + 1.5q\right]_0^{175}$$

$$= 10938. - \left(\frac{.002(175)^3}{3} + 1.5(175)\right) + 0 = \boxed{7102.6}$$

$\therefore$ total gains from trade = $\$7655.8 + \$7102.6 = \boxed{\$14758.}$

3. When the price of a certain product is $40, 25 items can be sold. When the price of the same product costs $20, 185 items can be sold. On the other hand, when the price of this product is $40, 200 items will be produced. But when the price of this product is $20, only 100 items will be produced. Use this information to find supply and demand functions (assume for simplicity that the functions are linear), and compute the consumer and producer surplus at the equilibrium price.

Let $p =$ price and let $q =$ quantity demanded at price p
The supply function is $s(q) = aq + b$, where a and b are constants to be determined.
The demand function is $d(q) = cq + e$, where c and e are constants to be determined.

$p = 40 \Rightarrow q = 25$
$p = 20 \Rightarrow q = 185$
Thus, the points $(25, 40)$ and $(185, 20)$ will determine the line given by $p = d(q)$.

$$p - 40 = \frac{20 - 40}{185 - 25}(q - 25)$$

$$\Rightarrow p - 40 = -\frac{1}{8}(q - 25)$$

$$\Rightarrow p = -\frac{q}{8} + \frac{25}{8} + 40$$

$\therefore \boxed{p = d(q) = -\frac{q}{8} + \frac{345}{8}}$ is the demand function.

$p = 40 \Rightarrow q = 200$
$p = 20 \Rightarrow q = 100$
Thus, the points $(200, 40)$ and $(100, 20)$ will determine the line given by $p = s(q)$.

$$p - 40 = \frac{20 - 40}{100 - 200}(q - 200)$$

$$\Rightarrow p - 40 = \frac{1}{5}(q - 200)$$

$$\Rightarrow p = \frac{q}{5} - \frac{200}{5} + 40 = \frac{q}{5} - 40 + 40 =$$

$\therefore \boxed{p = s(q) = \frac{q}{5}}$ is the supply function.

The equilibrium price (q^*, p^*) is the intersection point of the demand curve and the supply curve (in this case two straight lines).

$$-\frac{q}{8} + \frac{345}{8} = \frac{q}{5}$$

$$\Rightarrow \frac{q}{5} + \frac{q}{8} = \frac{345}{8}$$

The equilibrium price (q^*, p^*) is the intersection point of the demand curve and the supply curve (in this case two straight lines).

$$-\frac{q}{8} + \frac{345}{8} = \frac{q}{5}$$

$$\Rightarrow \frac{q}{5} + \frac{q}{8} = \frac{345}{8}$$

$$\Rightarrow (40)\frac{q}{5} + (40)\frac{q}{8} = (40)\frac{345}{8}$$

$$\Rightarrow 8q + 5q = (5)(345)$$

$$\Rightarrow 13q = (5)(345)$$

$$\Rightarrow q = \frac{(5)(345)}{13} = \frac{1725}{13} \approx 133$$

$$\Rightarrow p = \frac{(5)(345)}{(13)(5)} = \frac{345}{13} \approx 26.54$$

$\therefore$ the equilibrium point is $\boxed{(q^*, p^*) = (133, 26.54)}$

consumer surplus = $\int_0^{q^*} d(q)dq - p^*q^* = \int_0^{133}\left(-\frac{q}{8} + \frac{345}{8}\right)dq - (26.54)(133)$

$$= \left[-\frac{q^2}{16} + \frac{345q}{8}\right]_0^{133} - (26.54)(133)$$

$$= \left(-\frac{133^2}{16} + \frac{345(133)}{8}\right) - 0 - (26.54)(133) = 1100.2$$

$\therefore$ $\boxed{\text{consumer surplus} = \$1100.2}$

producer surplus = $p^*q^* - \int_0^{q^*} s(q)dq = (26.54)(133) - \int_0^{133} \frac{q}{5}dq$

$$= (26.54)(133) - \left[\frac{q^2}{10}\right]_0^{133}$$

$$= (26.54)(133) - \left(\frac{(133)^2}{10}\right) + 0 = 1760.9$$

$\therefore$ $\boxed{\text{producer surplus} = \$1760.9}$

producer surplus = $p^*q^* - \int_0^{q^*} s(q)dq = (26.54)(133) - \int_0^{133} \frac{q}{5}dq$

$$= (26.54)(133) - \left[\frac{q^2}{10}\right]_0^{133}$$

$$= (26.54)(133) - \left(\frac{(133)^2}{10}\right) + 0 = 1760.9$$

$\therefore$ producer surplus = $\$1760.9$

5. Find the present value of a continuous income stream of $40,000 per year for 35 years if money can earn
(a) 0.8% annual interest, compounded continuously,
(b) 2.5% annual interest, compounded continuously,
(c) 4.5% annual interest, compounded continuously.

Continuous Income Stream

Suppose money can earn interest at an annual interest rate of r, compounded continuously. Let F(t) be a continuous income function (in dollars per year) that applies between year 0 and year T.

Then the present value of that income stream is given by $PV = \int_0^T F(t)e^{-rt}\,dt$.

The future value can be computed by the ordinary compound interest formula $FV = PVe^{rt}$

(5a) $PV = \int_0^{35} 40000e^{-.008t}dt = 1.2211 \times 10^6 = \$1,221,100$

(5b) $PV = \int_0^{35} 40000e^{-.025t}dt = 9.3302 \times 10^5 = \$933,020$

(5c) $PV = \int_0^{35} 25000e^{-.034t}dt = 5.116 \times 10^5 = \$511,600$

Here, I carried out the calculations in Scientific Notebook 5.5.

7. Find the present value of a continuous income stream $F(t) = 12 + 0.3t^2$, where t is in years and F is in thousands of dollars per year, for 8 years, if money can earn 3.7% annual interest.

7. Find the present value of a continuous income stream $F(t) = 12 + 0.3t^2$, where t is in years and F is in thousands of dollars per year, for 8 years, if money can earn 3.7% annual interest, compounded continuously.

$$\int_0^8 (12 + 0.3t^2)e^{-.037t}\,dt = 124.17$$

$$\Rightarrow PV = \$124,170$$

Here, I carried out the calculations in Scientific Notebook 5.5.

9. A business is expected to generate income at a continuous rate of $25,000 per year for the next eight years. Money can earn 3.4% annual interest, compounded continuously. The business is for sale for $153,000. Is this a good deal?

$$PV = \int_0^8 25000e^{-.034t}\,dt = 1.7511 \times 10^5 = \$175,110 > \$153,000$$

Because the present value of the business is greater than the sales price, this is a good deal for the buyer.

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