2-9 Applied Calculus Solutions

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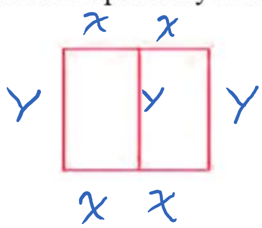
1. (a) You have 200 feet of fencing available to construct a rectangular pen with a fence divider down the middle (see below). What dimensions of the pen enclose the largest total area?
(b) If you need 2 dividers, what dimensions of the pen enclose the largest area?
(c) What are the dimensions in parts (a) and (b) if one edge of the pen borders on a river and does not require any fencing?



(1a)

Let x = width of each rectangle, in feet.
Let y = height each rectangle, in feet
Let A = total enclosed area, in square feet.

$A = (2x)y$

We want to maximize A, but we need A to be a function of a single variable.

total length of fencing = $200\text{ft} = 4x + 3y$

$\Rightarrow y = \dfrac{200 - 4x}{3}$

$\Rightarrow A(x) = (2x)\left(\dfrac{200 - 4x}{3}\right)$

$\therefore \boxed{A(x) = \dfrac{400}{3}x - \dfrac{8}{3}x^2}$

The total length of fencing constrains x to satisfy $0 \le 4x \le 200 \Rightarrow 0 \le x \le 50$

$$A'(x) = \frac{-16x}{3} + \frac{400}{3} = 0$$

$$\Rightarrow -16x + 400 = 0$$
$$\Rightarrow 16x = 400$$
$$\Rightarrow x = \frac{400}{16} = 25$$
$$\Rightarrow y = \frac{200 - 4(25)}{3} = \frac{100}{3}$$

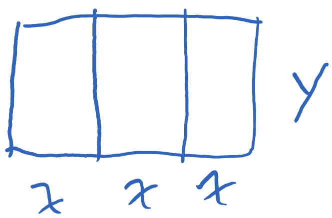
This is the only critical point, but we should still verify that the area is maximized.
We could use the first or second derivative test,
but on a closed interval, we need only compare the value at the critical point with the values at the endpoints.

$$A(25) = 2(25\,\text{ft})\left(\frac{100}{3}\,\text{ft}\right) = \frac{5000}{3}\,\text{ft}^2 \approx \boxed{1666.7\,\text{ft}^2}$$
$$A(0) = 2(0)\left(\frac{200 - 4(0)}{3}\right) = \boxed{0\,\text{ft}^2}$$

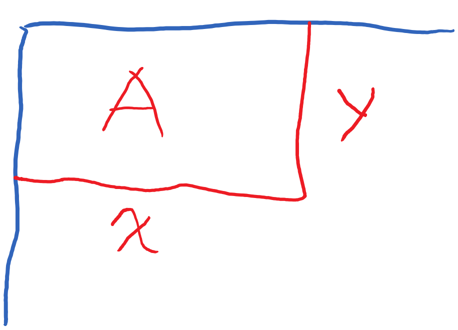
$$A(50) = 2(50)\left(\frac{200 - 4(50)}{3}\right) = 2(50)\left(\frac{200 - 200}{3}\right) = \boxed{0\,\text{ft}^2}$$

The dimensions of the pen are $2x$ by y,
or $\boxed{50\,\text{ft} \times \frac{100}{3}\,\text{ft}}$

(1b) With 2 dividers, we use the same approach.



3. Suppose you decide to fence the rectangular garden in the corner of your yard. Then two sides of the garden are bounded by the yard fence which is already there, so you only need to use the 80 feet of fencing to enclose the other two sides. What are the dimensions of the new garden of largest area? What are the dimensions of the rectangular garden of largest area in the corner of the yard if you have F feet of new fencing available?



Let x = base of rectangle
Let y = height of rectangle
Let A = area of rectangle

$2x + 2y = 80\text{ft}$
$A = xy$
We must maximize A.

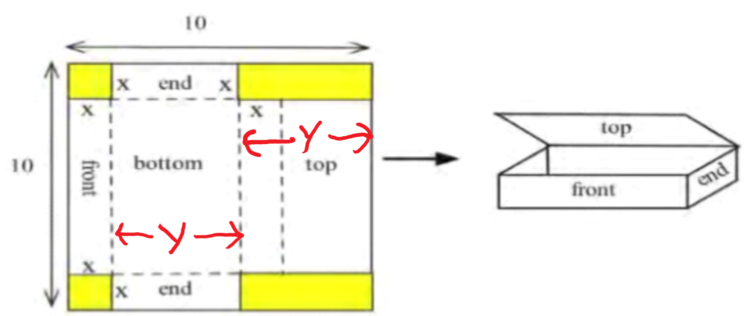
$y = \dfrac{80 - 2x}{2} = 40 - x$
$\Rightarrow A(x) = x(40 - x)$
The total length of fencing $\Rightarrow 0 \le 2x \le 80 \Rightarrow 0 \le x \le 40$

$A(x) = 40x - x^2$
$A'(x) = 40 - 2x = 0$
$\Rightarrow 2x = 40$
$\Rightarrow \boxed{x = 20\text{ft}}$
$y = 40 - 20 = 20\text{ft}$
We need only compare the areas at the critical point and at the endpoints of the interval.

$A(20) = (20\text{ft})(40\text{ft} - 20\text{ft}) = \boxed{400\text{ft}^2}$
$A(0) = (0\text{ft})(40\text{ft} - 0\text{ft}) = \boxed{0\text{ft}^2}$
$A(40) = (40\text{ft})(40\text{ft} - 40\text{ft}) = \boxed{0\text{ft}^2}$
$\therefore$ the dimensions of the gardent are $\boxed{20\text{ft} \times 20\text{ft}}$

With F feet of fencing, we can go through the same calculations as above, replacing the the equation $2x + 2y = 80$ with $2x + 2y = F$
You should obtain a square garden of dimensions $\left(\dfrac{F\text{ft}}{4}\right) \times \left(\dfrac{F\text{ft}}{4}\right)$.

5. You have a 10 inch by 10 inch piece of cardboard which you plan to cut and fold as shown to form a box with a top. Find the dimensions of the box which has the largest volume.



Let V = volume of box
Let x = height of box = side of square corner
Let y = length of end

$V = $ (length of front)(length of end)(height)

$V = (10-2x)(y)(x)$

$x + 2y = 10 \Rightarrow 2y = 10 - x \Rightarrow \boxed{y = \frac{10-x}{2}}$

$V = (10-2x)\left(\frac{10-x}{2}\right)(x) = (5-x)(10-x)x$

$\boxed{V = x^3 - 15x^2 + 50x}$

$0 \le 2x \le 10 \Rightarrow \boxed{0 \le x \le 5}$

We need to find any critical points and compare them with the endpoints.

$V\left(\frac{15-5\sqrt{3}}{3}\right) = \frac{250}{9}\sqrt{3} \approx \boxed{48.113}$

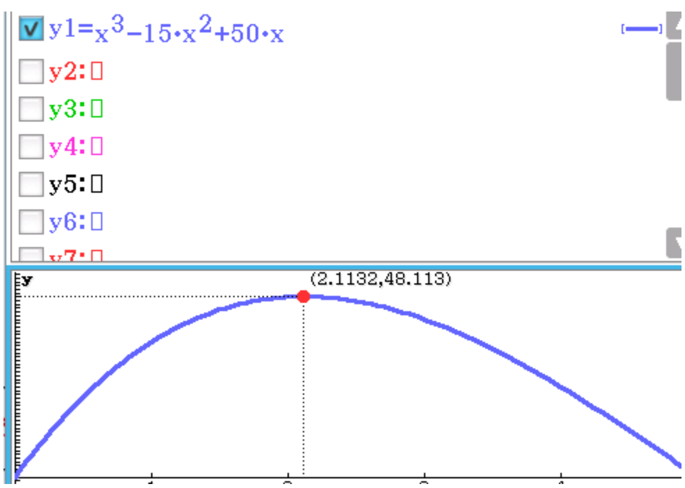
$V(0) = 0^3 - 15(0)^2 + 50(0) = \boxed{0}$

$V(5) = 5^3 - 15(5)^2 + 50(5) = \boxed{0}$

$\therefore \boxed{x = \frac{15-5\sqrt{3}}{3} \text{ in} \approx 2.1132 \text{ in}}$

$$\boxed{y = \frac{10 - \frac{15-5\sqrt{3}}{3}}{2} = \left(\frac{5}{6}\sqrt{3} + \frac{5}{2}\right)\text{in} \approx 3.9434 \text{ in}}$$

$$\boxed{10 - 2x = 10 - 2\left(\frac{15-5\sqrt{3}}{3}\right) = \frac{10}{3}\sqrt{3} \text{ in} = 5.7735 \text{ in}}$$



7. (a) Determine the dimensions of the least expensive cylindrical can which will hold 100 cubic inches if the materials cost 2¢, 5¢ and 3¢ respectively for the top, bottom and sides.
(b) How do the dimensions of the least expensive can change if the bottom material costs more than 5¢ per square inch?

(7a) The instructions must be clarified. The cost of materials are respectively $\frac{2¢}{\text{in}^2}, \frac{5¢}{\text{in}^2}, \frac{3¢}{\text{in}^2}$ for the top, bottom, and sides.

Let $r =$ radius of can, in inches
Let $h =$ height of can, in inches

Let $V =$ volume of can

$V = \pi r^2 h = 100\,\text{in}^3 \Rightarrow h = \frac{100\,\text{in}^3}{\pi r^2}$
Area of top $= \pi r^2 \Rightarrow$ cost of top $= 2\pi r^2$
Area of bottom $= \pi r^2 \Rightarrow$ cost of bottom $= 5\pi r^2$
Area of side $=$ (circumference)(height) $= (2\pi r)h \Rightarrow$ cost of side $= 6\pi r h$

Let $C =$ total cost of materials
$C = 2\pi r^2 + 5\pi r^2 + 6\pi r h$

$C(r) = 2\pi r^2 + 5\pi r^2 + 6\pi r\left(\frac{100}{\pi r^2}\right) = 7\pi r^2 + \frac{600}{r}$
We must minimize $C(r)$.
Because $V > 0$, we must have $r > 0$, so $C(r)$ is defined for $r > 0$.

$C(r) = 7\pi r^2 + 600r^{-1}$
$\Rightarrow C'(r) = 14\pi r - 300r^{-2} = 0$

$\Rightarrow 7\pi r = \frac{300}{r^2} \Rightarrow r^3 = \frac{300}{7\pi}$

$$r = \sqrt[3]{\frac{300}{7\pi}}\ \text{in} \approx 2.3894\,\text{in}$$

$$h = \frac{100}{\pi\left(\sqrt[3]{\frac{300}{7\pi}}\right)^2}\ \text{in} \approx 5.5753\,\text{in}$$

We can use the second derivative test to prove this is a minimum.

$$C''(r) = \frac{d}{dr}\left(14\pi r - 300r^{-2}\right) = 14\pi r - 600r^{-2}$$

$$\Rightarrow C''\left(14\pi\left(\frac{100}{\pi\left(\sqrt[3]{\frac{300}{7\pi}}\right)^2}\right) - 600\left(\frac{100}{\pi\left(\sqrt[3]{\frac{300}{7\pi}}\right)^2}\right)^{-2}\right) \approx 225.91 > 0$$

$\therefore$ This is a local minimum.

Since there are no other critical points or endpoints,
this is also a global minimum.

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These solutions were created by Donald R. Goral from exercises in *Applied Calculus, Edition 1*
by Shana Calaway, Dale Hoffman, David Lippman

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