04-03-25 MDE 161

3.3 Real Zeros of Polynomials
    3.3.3: Exercises
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3.4 Complex Zeros and the Fundamental Theorem of Algebra
    3.4.1 Exercises
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Memorize

Definition 3.4. The imaginary unit i satisfies the two following properties

1. $i^2 = -1$
2. If c is a real number with $c \geq 0$ then $\sqrt{-c} = i\sqrt{c}$

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

$$c \geq 0, \quad \sqrt{-c} = \sqrt{c}\sqrt{-1} = (\sqrt{c})(i) = i\sqrt{c}$$

Memorize

Definition 3.5. A **complex number** is a number of the form $a + bi$, where a and b are real numbers and i is the imaginary unit.

$\mathbb{C} = \{a + bi | a \in \mathbb{R}, b \in \mathbb{R}, i^2 = -1\}$

$\mathbb{C}, \mathbb{R}$

$$(2+3i) + (5-i) = (2+5) + (3-1)i$$
$$= \boxed{7 + 2i}$$

Memorize

Def $(a+bi) \pm (c+di)$
$$= (a+c) \pm (b+d)i$$

$$(2+3i)(5-i)$$

$$\boxed{i^2 = -1}$$

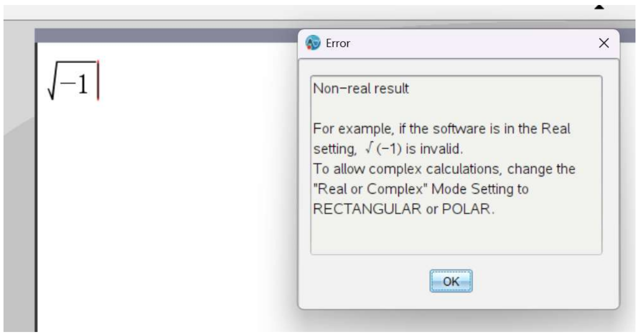
$$= (2)(5) + (3i)(5) + (2)(-i) + (3i)(-i)$$

$$= (2)(5) + (3i)(5) + (2)(-i) + (3i)(-i)$$

$$= 10 + 15i - 2i - 3i^2$$

$$= 10 + 13i - 3(-1)$$

$$= \boxed{13 + 13i}$$



Memorize

Definition:

Let $z = a + bi \in \mathbb{C}$

Then $\bar{z}$ = conjugate of z = $a - bi \in \mathbb{C}$

$$(13 + 13i) \div (5 - i)$$

$$\frac{13 + 13i}{5 - i} = \left(\frac{13 + 13i}{5 - i}\right)\left(\frac{5 + i}{5 + i}\right)$$

$$\boxed{(a-b)(a+b) = a^2 - b^2}$$

$$= \frac{(13 + 13i)(5 + i)}{(5 - i)(5 + i)}$$

$$= \frac{[(13)(5) + 13i^2] + [(5)(13) + 13]\,i}{5^2 - i^2}$$

$$= (65 - 13) + (65 + 13)\,i$$

$$= \frac{(65-13)+(65+13)i}{25-(-1)}$$

$$= \frac{52+78i}{26}$$

$$= \boxed{2+3i}$$

$$\frac{1}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{\sqrt{2}}{2}$$

"rationalizing the denominator"

Understand

Theorem 3.12. Properties of the Complex Conjugate: Let z and w be complex numbers.

- $\overline{\overline{z}} = z$
- $\overline{z} + \overline{w} = \overline{z+w}$
- $\overline{z}\,\overline{w} = \overline{zw}$
- $(\overline{z})^n = \overline{z^n}$, for any natural number n
- z is a real number if and only if $\overline{z} = z$.

Let $z = a+bi$

$\Rightarrow \overline{z} = \overline{a+bi} = a-bi$

$\Rightarrow \overline{\overline{z}} = \overline{\overline{a+bi}} = \overline{a-bi}$

$= a+bi = z$

$\therefore \overline{\overline{z}} = z$

supplied

Theorem 3.13. The Fundamental Theorem of Algebra: Suppose f is a polynomial function with complex number coefficients of degree $n \geq 1$, then f has at least one complex zero.

Supplied

Theorem 3.14. Complex Factorization Theorem: Suppose f is a polynomial function with complex number coefficients. If the degree of f is n and $n \geq 1$, then f has exactly n complex zeros, counting multiplicity. If $z_1, z_2, \ldots, z_k$ are the distinct zeros of f, with multiplicities $m_1, m_2, \ldots, m_k$, respectively, then $f(x) = a\,(x - z_1)^{m_1}\,(x - z_2)^{m_2} \cdots (x - z_k)^{m_k}$.

Memorize

Theorem 3.15. Conjugate Pairs Theorem: If f is a polynomial function with real number coefficients and z is a zero of f, then so is $\overline{z}$.

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$b^2 - 4ac > 0$

$x = \frac{-b \pm \text{real number}}{2a}$ 2 real distinct solutions

$b^2 - 4ac = 0$ $x = \frac{-b \pm 0}{2a} = \frac{-b}{2a}$ 1 real solution mult 2

$b^2 - 4ac < 0$

$x = \frac{-b \pm (?)i}{2a}$ 2 complex conjugate solutions

Memorize

Theorem 3.16. Real Factorization Theorem: Suppose f is a polynomial function with real number coefficients. Then $f(x)$ can be factored into a product of linear factors corresponding to the real zeros of f and irreducible quadratic factors which give the nonreal zeros of f.

3.4

In Exercises 11 - 18, simplify the quantity.

13. $\sqrt{-25}\sqrt{-4}$

$\sqrt{25}\sqrt{-1}\,\sqrt{4}\,\sqrt{-1}$

14. $\sqrt{(-25)(-4)}$

$= \sqrt{100}$

$\sqrt{25}\sqrt{-1}\;\sqrt{4}\;\sqrt{-1}$

$(5i)(2)i$

$= 10i^2$

$= 10(-1)$

$= \boxed{-10}$

$= \sqrt{100}$

$= 10$

$\sqrt{-25}\cdot\sqrt{-4}$ Warning: Non-real calculation   OK   -10

$\sqrt{-25\cdot -4}$ 10

Conjecture.

$$\sqrt{(a+bi)(c+di)} = \sqrt{a+bi}\;\sqrt{c+di}$$

Think about this

3.4

In Exercises 49 - 53, create a polynomial f with real number coefficients which has all of the desired characteristics. You may leave the polynomial in factored form.

52.
- f is degree 5.
- $x = 6$, $x = i$ and $x = 1 - 3i$ are zeros of f
- as $x \to -\infty$, $f(x) \to \infty$

f is of deg 5 $\Rightarrow$ leading term ax^5, $a \in \mathbb{R}$

$x = 6$ is a zero $\Rightarrow$ $x - 6$ is a factor of $f(x)$

$x = i$ is a zero $\Rightarrow$ $x - i$ is a factor

$\Rightarrow$ $x = -i$ is a zero

$\Rightarrow$ $x - (-i)$ is a factor

$x = 1 - 3i$ is a zero $\Rightarrow$ $x = 1 + 3i$ is a zero

$\Rightarrow$ $x - (1 - 3i)$ is a factor of $f(x)$

$\Rightarrow$ $x-(1-3i)$ is a factor of $f(x)$

$x-(1+3i)$ is " " " "

$$\boxed{f(x) = a(x-6)(x+i)(x-i)((x-1)+3i)((x-1)+3i)}$$

$a<0$

leading term ax^5

end behavior $x \to -\infty \Rightarrow f(x) \to \infty$

$\Rightarrow a<0$