


5.4 Integration by Substitution
page 160: 1, 3, 5, 15, 23, 27

7 Analytic Geometry and Plain Curves
7.5 Hyperbolic Functions
page 246: 1 (for $\sinh(-x)$), 12, 21, 27

SN omitted the necessary absolute value.

$$\int \frac{1}{x} dx = \ln x'$$

TI-Nspire did insert the absolute value sign.



The image shows a TI-Nspire calculator screen. On the left, the integral $\int \frac{1}{x} dx$ is displayed. On the right, the result $\ln(|x|)$ is shown.

However, each program requires the user to insert the arbitrary constant C .

What is the integral of $1/x$?

◆ The integral of $\frac{1}{x}$ with respect to x is:

⊙
$$\int \frac{1}{x} dx = \ln |x| + C$$

Key Components of the Result:

- Natural Logarithm (\ln):** The function \ln represents the logarithm to the base e (approximately 2.718).
- Absolute Value ($|x|$):** The absolute value is necessary because the domain of the natural logarithm is restricted to positive numbers ($x > 0$), whereas the function $\frac{1}{x}$ is defined for all real numbers except $x = 0$. Using $|x|$ ensures the integral is valid for both positive and negative values of x .
- Constant of Integration (C):** Since this is an indefinite integral, we add C to represent any constant that would disappear during differentiation.

Why the Power Rule Doesn't Apply

Usually, we use the power rule for integration:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

However, if you try to apply this to $\frac{1}{x}$ (which is x^{-1}), you would get:

$$\int x^{-1} dx = \frac{x^0}{0}$$

Division by zero is undefined, which is why $\frac{1}{x}$ is the unique case where the power rule fails and a logarithmic function is required.

B

27. Recall from trigonometry that $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ for all x .

(a) Use the Fundamental Theorem of Calculus to evaluate $\int_0^\pi \sin^2 x dx$. = \mathcal{I}

(b) Approximate the integral from part (a) by dividing the interval $[0, \pi]$ into $n = 2$ subintervals of equal length, $[0, \pi/2]$ and $[\pi/2, \pi]$, and finding the **exact** value of the sum of the areas of the rectangles whose heights are determined at the right endpoints of the subintervals.

(c) Repeat part (b) with $n = 3$.

(d) Repeat part (b) with $n = 4$.

(e) Repeat part (b) with $n = 6$.

$$\mathcal{I} = \frac{1}{2} \int_0^\pi (1 - \cos 2x) dx$$

$$= \frac{1}{2} \int_0^\pi dx - \frac{1}{2} \int_0^\pi \cos 2x dx$$

$$= \left[\frac{x}{2} \right]_0^\pi - \left(\frac{1}{2} \right) \mathcal{I}_2$$

$$\mathcal{I}_2 = \int_0^\pi \cos 2x dx$$

$$\text{Let } u = 2x$$

$$\Rightarrow du = 2 dx \Rightarrow dx = \frac{du}{2}$$

$$x = 0 \Rightarrow u = (2)(0) = 0$$

$$x = \pi \Rightarrow u = 2\pi$$

$$x=0 \Rightarrow u = (2\pi/0) \dots$$

$$x=\pi \Rightarrow u = 2\pi$$

$$I_2 = \frac{1}{2} \int_0^{2\pi} \cos u \, du$$

$$= \frac{1}{2} \left[\sin u \right]_0^{2\pi}$$

$$= \frac{1}{2} \left[\sin(2\pi) - \sin(0) \right]$$

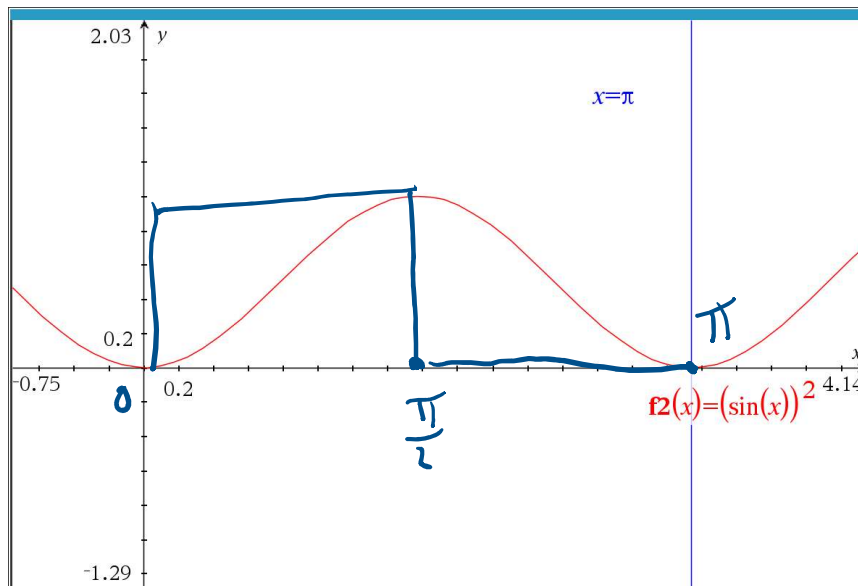
$$= \frac{1}{2} (0 - 0) = 0$$

$$\boxed{I_2 = 0}$$

$$I = \left[\frac{x}{2} \right]_0^{\pi} - \left(\frac{1}{2} \right) (0)$$

$$I = \frac{\pi}{2} - \frac{0}{2}$$

$$\boxed{I = \frac{\pi}{2}}$$



$$\text{let } n=2$$

let $n=2$

$$(b) \int_0^{\pi} \sin^2 x \, dx \approx \left(\frac{\pi}{2}\right)(1) + 0$$

$$\approx \frac{\pi}{2}$$

Scientific Notebook

$(\sin(x))^2$ Approximate integral (right boxes) is $1.57 \sum_{i_3=1}^2 \sin^2 1.57 i_3 = 1.57$

$(\sin(x))^2$ Approximate integral (right boxes) is $1.0467 \sum_{i_4=1}^3 \sin^2 1.0467 i_4 = 1.5705$

$(\sin(x))^2$ Approximate integral (right boxes) is $0.785 \sum_{i_5=1}^4 \sin^2 0.785 i_5 = 1.5706$

$(\sin(x))^2$ Approximate integral (right boxes) is $0.52333 \sum_{i_6=1}^6 \sin^2 0.52333 i_6 = 1.5707$

$(\sin(x))^2$ Approximate integral (right boxes) is $0.0314 \sum_{i_7=1}^{100} \sin^2 0.0314 i_7 = 1.5708$

$(\sin(x))^2$ Approximate integral (right boxes) is $0.00314 \sum_{i_8=1}^{1000} \sin^2 0.00314 i_8 = 1.5708$

$$\int_0^{3.14} (\sin(x))^2 dx = 1.5708$$

7.5

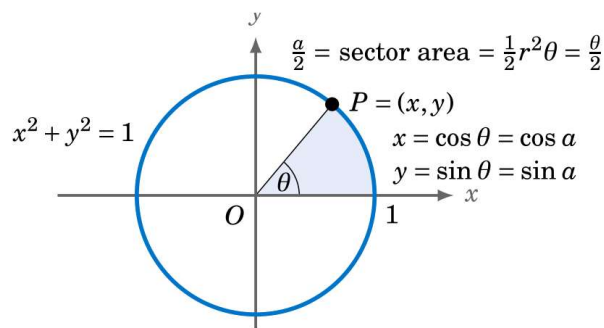


Figure 7.5.1 Circular

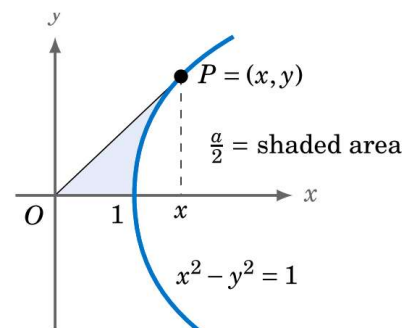


Figure 7.5.2 Hyperbolic

Read the textbook explanation.

Let $A = \text{area of sector}$

$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$

$$A = \left(\frac{1}{2}\right) \theta r^2$$

$$r=1 \Rightarrow A = \theta$$

$$r=1 \Rightarrow A = \frac{\theta}{2}$$

define $\frac{a}{2} = \text{sector area}$

supplied

The **hyperbolic sine**, **hyperbolic cosine**, **hyperbolic tangent**, **hyperbolic cotangent**, **hyperbolic secant** and **hyperbolic cosecant**, denoted by \sinh , \cosh , \tanh , \coth , sech and csch , respectively, are:

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{for all } x$$

$$\tanh x = \frac{\sinh x}{\cosh x} \quad \text{for all } x$$

$$\operatorname{sech} x = \frac{1}{\cosh x} \quad \text{for all } x$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{for all } x$$

$$\coth x = \frac{1}{\tanh x} \quad \text{for all } x \neq 0$$

$$\operatorname{csch} x = \frac{1}{\sinh x} \quad \text{for all } x \neq 0$$

The other formulas are analogous to trig functions.

Supplied or I will ask you to derive from the above formulas.

$$\cosh^2 x - \sinh^2 x = 1$$

$$\tanh^2 x + \operatorname{sech}^2 x = 1$$

$$\coth^2 x - \operatorname{csch}^2 x = 1$$

$$\sinh(-x) = -\sinh x$$

$$\cosh(-x) = \cosh x$$

$$\tanh(-x) = -\tanh x$$

$$\operatorname{sech}(-x) = \operatorname{sech} x$$

$$\operatorname{csch}(-x) = -\operatorname{csch} x$$

$$\coth(-x) = -\coth x$$

$$\sinh(u \pm v) = \sinh u \cosh v \pm \cosh u \sinh v$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh(u \pm v) = \cosh u \cosh v \pm \sinh u \sinh v$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\tanh(u \pm v) = \frac{\tanh u \pm \tanh v}{1 \pm \tanh u \tanh v}$$

$$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$$

Supplied or I will ask you to derive from the basic formulas.

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

$$\int \cosh x \, dx = \sinh x + C$$

$$\int \sinh x \, dx = \cosh x + C$$

$$\int \operatorname{sech}^2 x \, dx = \tanh x + C$$

$$\int \operatorname{csch} x \coth x \, dx = -\operatorname{csch} x + C$$

$$\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + C$$

$$\int \operatorname{csch}^2 x \, dx = -\coth x + C$$

Supplied or I ask you to prove.

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$$

$$\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x} \quad \text{for } |x| < 1$$

$$\operatorname{sech}^{-1} x = \ln \frac{1 + \sqrt{1-x^2}}{x} \quad \text{for } 0 < x \leq 1$$

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \quad \text{for } x \geq 1$$

$$\coth^{-1} x = \frac{1}{2} \ln \frac{x+1}{x-1} \quad \text{for } |x| > 1$$

$$\operatorname{csch}^{-1} x = \ln \left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1} \right) \quad \text{for } x \neq 0$$

Supplied or I ask you to prove.

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2 + 1}}$$

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2} \quad \text{for } |x| < 1$$

$$\frac{d}{dx}(\operatorname{sech}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}} \quad \text{for } 0 < x \leq 1$$

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}} \quad \text{for } x \geq 1$$

$$\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2} \quad \text{for } |x| > 1$$

$$\frac{d}{dx}(\operatorname{csch}^{-1} x) = \frac{-1}{|x|\sqrt{1+x^2}} \quad \text{for } x \neq 0$$

7.5

A

A

1. Prove the identities for $\sinh(-x)$, $\cosh(-x)$, $\tanh(-x)$, $\coth(-x)$, $\operatorname{sech}(-x)$, and $\operatorname{csch}(-x)$ on p.232.

$$\cosh(-x) = \cosh x$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{for all } x$$

$$\begin{aligned}\cosh(-x) &= \frac{e^{-x} + e^{-(-x)}}{2} \\ &= \frac{e^{-x} + e^x}{2} \\ &= \frac{e^x + e^{-x}}{2} \\ &= \cosh(x)\end{aligned}$$

10. $\coth^{-1}\left(\frac{1}{x}\right) = \tanh^{-1}x$

$$\coth^{-1}x = \frac{1}{2} \ln \frac{x+1}{x-1} \quad \text{for } |x| > 1$$

$$\tanh^{-1}x = \frac{1}{2} \ln \frac{1+x}{1-x} \quad \text{for } |x| < 1$$

$$\begin{aligned}\coth^{-1}\left(\frac{1}{x}\right) &= \frac{1}{2} \ln \left(\frac{\frac{1}{x} + 1}{\frac{1}{x} - 1} \right) \quad \text{for } \left| \frac{1}{x} \right| > 1 \\ &= \frac{1}{2} \ln \left(\frac{\frac{1}{x} + \frac{x}{x}}{\frac{1}{x} - \frac{x}{x}} \right) \quad \begin{array}{l} \frac{1}{|x|} > 1 \\ \frac{1}{|x|} > 1 \\ \frac{1}{|x|} > 1 \\ |x| > 1 \\ |x| < 1 \end{array} \\ &= \frac{1}{2} \ln \left(\frac{\frac{x+1}{x}}{\frac{1-x}{x}} \right)\end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) \\
 &= \frac{1}{2} \ln \left(\frac{x+1}{1-x} \right) \\
 &= \tanh^{-1}(x) \text{ for } |x| < 1
 \end{aligned}$$

What practical applications do hyperbolic functions have?

◆ Hyperbolic functions—primarily $\sinh(x)$, $\cosh(x)$, and $\tanh(x)$ —frequently appear in physics and engineering because they describe shapes and motions governed by exponential growth or decay in opposing directions.

Here are some of the most prominent practical applications:

1. The Catenary Curve (Power Lines and Chains)

The most famous application of the hyperbolic cosine function (\cosh) is the **catenary**. When a flexible cable, chain, or rope is suspended between two points and hangs under its own weight, it forms a catenary curve.

- **Application:** Structural engineers use the equation $y = a \cosh\left(\frac{x}{a}\right)$ to calculate the tension and sag in overhead power lines, suspension bridge cables (before the deck is added), and even the shape of the **Gateway Arch** in St. Louis, which is an inverted catenary.

2. Special Relativity

In physics, hyperbolic functions are essential for understanding space-time. Because the "distance" in Minkowski space-time involves a subtraction ($s^2 = c^2t^2 - x^2$) rather than the addition used in Euclidean geometry, rotations in space-time (Lorentz transformations) are hyperbolic.

- **Rapidity:** Instead of using standard velocity, physicists often use **rapidity** (ϕ), where velocity $v = c \tanh(\phi)$. This makes calculating the addition of near-light speeds much simpler, as rapidities simply add together linearly.

3. Fluid Dynamics and Oceanography

Hyperbolic functions are used to model the motion of waves in water, particularly in "intermediate" depths where the water is neither very shallow nor very deep.

- **Wave Velocity:** The speed of a gravity wave in water is often expressed using \tanh .
- **Vertical Motion:** The \sinh and \cosh functions describe how the pressure and the orbital motion of water particles decrease as you move from the surface down toward the seabed.

4. Heat Transfer and Material Science

When studying how heat moves through a long, thin object (like a cooling fin on a radiator or a computer chip), the temperature distribution along that object is often modeled using hyperbolic functions.

- **Fin Efficiency:** Engineers use \tanh to determine the efficiency of heat sinks. The "fin effectiveness" formula typically involves the hyperbolic tangent of the fin's geometric and thermal properties.

5. Neural Networks (Machine Learning)

In the digital world, the hyperbolic tangent (\tanh) is a staple in artificial intelligence.

- **Activation Functions:** $\tanh(x)$ is used as an "activation function" in neural networks. It maps any input value to a range between -1 and 1 . This is often preferred over the standard sigmoid function because it is "zero-centered," which helps the model learn more efficiently during the training process.

6. Mercator Projections (Map Making)

If you look at a standard rectangular world map, you are likely looking at a Mercator projection. The vertical spacing of the latitude lines on these maps is calculated using the inverse hyperbolic sine (arsinh) or related logarithmic functions to ensure that angles (and thus compass bearings) remain accurate for navigation.