04-08-26 MTH 263

5 The Integral
   5.1 The Indefinite Integral
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   5.2 The Definite Integral
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5.3 The Fundamental Theorem of Calculus
      page 155: 1, 4, 7, 11, 12, 13

5.2: 1

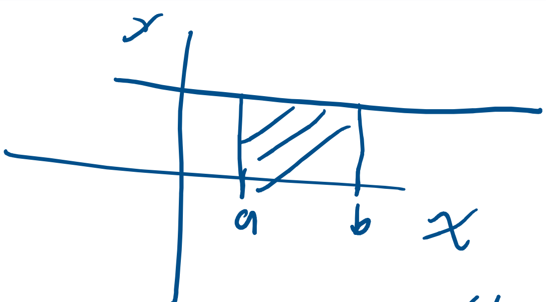
1. Explain why $\int_a^b c\,dx = c(b-a)$ for any constant c.

For a function $f(x) \geq 0$ over $[a,b]$, the **area under the curve** $y = f(x)$ between $x = a$ and $x = b$, denoted by A, is given by

$$A = \int_a^b f(x)\,dx$$

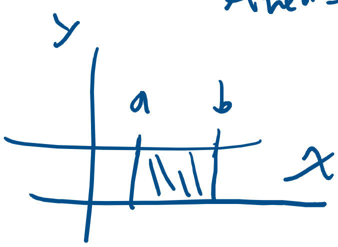
and represents the area of the region R bounded above by $y = f(x)$, bounded below by the x-axis, and bounded on the sides by $x = a$ and $x = b$ (with $a < b$).

Assume $c > 0$



$y = c > 0$

Area $= c(b-a)$ = area of rectangle



$y = c < 0$

Area $= c(b-a)$

Copilot

Could you please explain sigma notation?

Sigma notation is the compact language mathematicians use to express a sum whose terms follow a pattern. It replaces long, repetitive addition with a single symbolic instruction: "add these terms up according to this rule." [1] [2]

What the symbol means

- The symbol **Σ** is the uppercase Greek letter *sigma*, chosen because it corresponds to **S for "sum."**
- A typical expression looks like

$$\sum_{i=m}^{n} a_i$$

- and is read as "the sum of a_i from $i = m$ to $i = n$."
- The letter under the Σ is the **index of summation** (often i, j, or k).
- The numbers below and above the Σ are the **lower** and **upper bounds**—the starting and ending values of the index. [2]

How it works (mechanics)

To evaluate

$$\sum_{i=1}^{4} i,$$

you substitute each integer from 1 through 4 into the expression and add:

$$1 + 2 + 3 + 4 = 10.$$

This is exactly the same as writing the long sum, just more compact. [1]

You can place any expression after the Σ. For example:

- Squares: $\sum_{n=1}^{4} n^2 = 1^2 + 2^2 + 3^2 + 4^2$
- Even numbers: $\sum_{i=1}^{25} 2i = 2 + 4 + \ldots + 50$
- Patterned expressions: $\sum_{i=1}^{3} i(i+1) = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4$ [1] [3]

5.2: 7

7. Prove the formula $\sum_{k=1}^{n} k^2 = \dfrac{n(n+1)(2n+1)}{6}$ by induction on $n \geq 1$.

basis step

let $n=1$ $\qquad \sum_{k=1}^{1} k^2 \stackrel{?}{=} \dfrac{(1)(2)(3)}{6}$

Let $n=1$ $\quad \sum_{k=1}^{1} k^2 = \frac{1(1+1)(2+1)}{6}$

$$1^2 \stackrel{?}{=} \frac{6}{6}$$

$$1 = 1 \quad \checkmark$$

Assume true for any fixed $n \geq 1$ (Ind hyp)

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Prove $\quad \sum_{k=1}^{n+1} k^2 = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$

$$\boxed{\text{Prove} \quad \sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}}$$

$$\sum_{k=1}^{n+1} k^2 = \left(\sum_{k=1}^{n} k^2\right) + (n+1)^2$$

Ind. hyp $\quad = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)^2}{6}$$

$$= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6}$$

$$= \frac{(n+1)\left(n(2n+1) + 6(n+1)\right)}{6}$$

$$= \frac{(n+1)(2n^2 + n + 6n + 6)}{6}$$

$$= (n+1)(2n^2 + 7n + 6)$$

$$= \frac{(n+1)(2n^2+7n+6)}{6}$$

$$= \frac{(n+1)(n+2)(2n+3)}{6} \quad \checkmark$$

5.3
memorize

Fundamental Theorem of Calculus: Suppose that a function f is differentiable on $[a,b]$. Then:

(I) The function $A(x)$ defined on $[a,b]$ by

$$A(x) = \int_a^x f(t)\,dt$$

is differentiable on $[a,b]$, and

$$A'(x) = f(x)$$

for all x in $[a,b]$.

(II) If F is an antiderivative of f on $[a,b]$, i.e. $F'(x) = f(x)$ for all x in $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)\,.$$

$$\int_a^b c\,dx = cx\Big]_a^b = c(b-a)$$

$$\int_a^b k\,dx = kx + c\Big]_a^b$$

$$= (kb+c) - (ka+c)$$

$$= kb - ka$$

Therefore, **<u>never</u>** write an arbitrary constant for a definite integral.

To prove Part II of the theorem, let $F(x)$ be an antiderivative of $f(x)$ over $[a,b]$. Since $A(x) = \int_a^x f(x)\,dx$ is also an antiderivative of $f(x)$ over $[a,b]$ by Part I of the theorem, then $A(x)$ and $F(x)$ differ by a constant C over $[a,b]$. In other words:

$\int_a^x f(t)\,dt$, t = dummy variable

t cannot $= x$

Memorize

If f is an odd function, i.e. $f(-x) = -f(x)$ for all x, then

$$\int_{-a}^{a} f(x)\,dx = 0$$

for all $a > 0$ such that f is continuous on $[-a,a]$.

memorize

If f is an even function, i.e. $f(-x) = f(x)$ for all x, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

for all $a > 0$ such that f is continuous on $[-a,a]$.

Memorize

Let f and g be continuous functions on $[a,b]$ and let k be a constant. Then:

1. $\displaystyle\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$

2. $\displaystyle\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3. $\displaystyle\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Let f be a continuous function on $[a,b]$ and suppose that $a<c<b$. Then:

(1) $\displaystyle\int_a^a f(x)\,dx = 0$

(2) $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

(3) $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ (This works for $c \notin (a,b)$)

Let $F'(x) = f(x)$

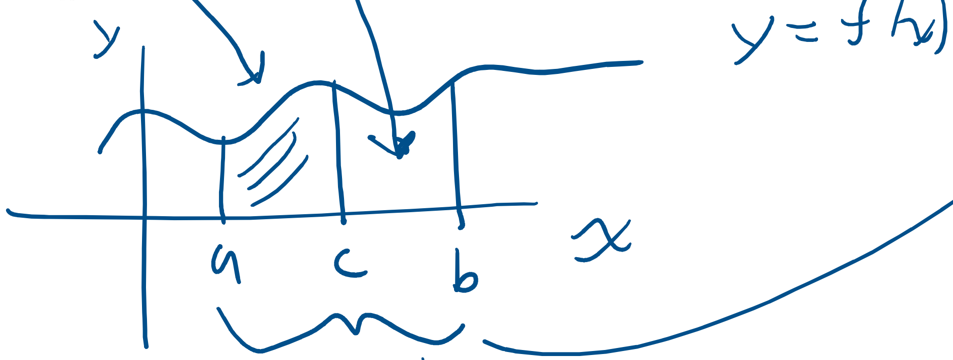
$$\int_a^a f(x)\,dx = F(x)\Big]_a^a = F(a) - F(a) = 0$$

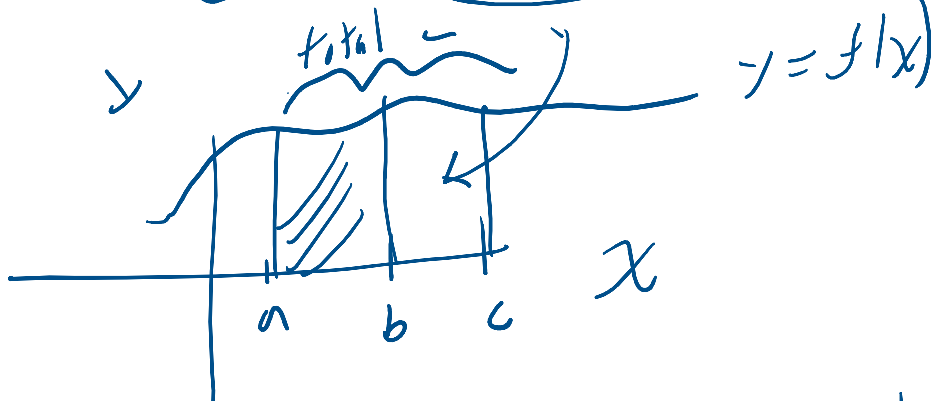
(3) $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

For example, if $F(x)$ is an antiderivative of $f(x)$ on $[a,b]$, then

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = (F(c)-F(a)) + (F(b)-F(c)) = F(b) - F(a) = \int_a^b f(x)\,dx$$

which proves rule (3).





$\int^b f(x)\,dx = \int^c \quad \int^b$

Prove $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx - \int_b^c f(x)\,dx$$

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$ which was proven in the text

$a < b < c$

Supplied

Chain Rule for integrals: Let f be a continuous function on an interval I containing $x = a$, and let $g(x)$ be a differentiable function on I. If

$$F(x) = \int_a^{g(x)} f(t)\,dt \quad \text{for all } x \text{ in } I$$

then $F'(x) = f(g(x)) \cdot g'(x)$ for all x in I.

Your name MTH 263 quiz 5

1. Evaluate $\int (2x + e^x)dx$. $= 2\frac{x^2}{2} + e^x = \boxed{x^2 + e^x + C}$

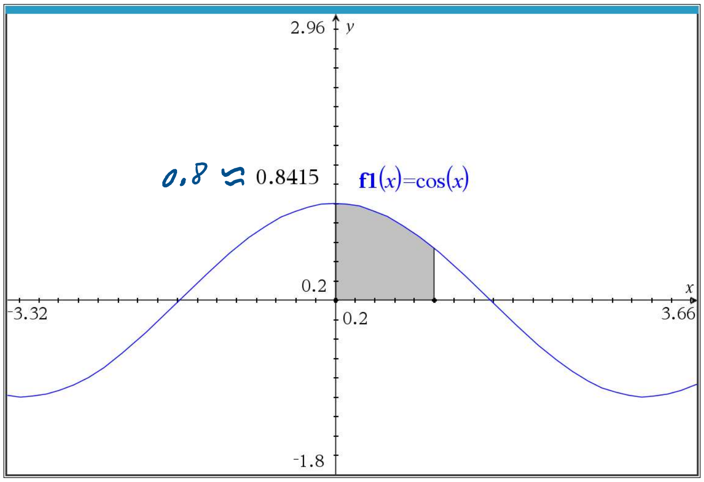
2. Evaluate $\int_0^1 \cos(x)dx$.
Give exact answer, then round to nearest tenth.

$$\int_0^1 \cos(x)\,dx = \sin(x)\Big]_0^1 = \sin(1) - \sin(0)$$

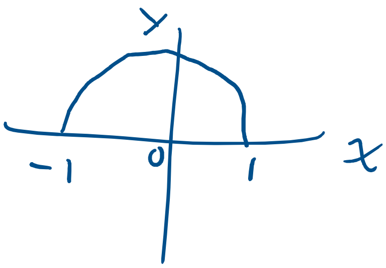
sin(1)   0.8414709848

$= \boxed{\sin(1) \approx 0.8}$

2. Evaluate $\int_0^1 \cos(x)dx$ on your calculator.
Sketch a labeled graph.
The answers agree.



4. Evaluate $\int_{-1}^{1} \sqrt{1-x^2}\,dx$. Hint: Use geometry.
   Give exact answer. Then round to nearest tenth.



$y = \sqrt{1-x^2}$

$y^2 = 1 - x^2$

$x^2 + y^2 = 1$ equation of unit circle

area of circle $= \pi r^2 = \pi(1^2) = \boxed{\pi}$

area of top semicircle $= \boxed{\frac{\pi}{2} \approx 1.6}$

Pi/2=1.570796326794897

$$\boxed{\int_{-1}^{1} \sqrt{1-x^2} = \frac{\pi}{2} \approx 1.6}$$

5. Evaluate $\int_{-1}^{1} \sqrt{1-x^2}\,dx$ on your calculator.
   Sketch a labeled graph.

The answers agree.