04-06-26 MTH 263

4.4 The Mean Value Theorem
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5 The Integral

5.1 The Indefinite Integral
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5.2 The Definite Integral
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Before class notes

4.4

18. Show that $f(x) = \cos^2 x + \cos^2\left(\frac{\pi}{3} + x\right) - \cos x \cos\left(\frac{\pi}{3} + x\right)$ is a constant function. What is its value?

> If f is a differentiable function on an interval I such that $f'(x) = 0$ for all x in I, then f is a constant function on I.

strategy: calculate f'(x) and show that it is equal to zero for all x.

$$\frac{d}{dx}\left(\cos^2 x\right) = \frac{d}{dx}\left(\left(\cos x\right)^2\right)$$

$$= 2\left(\cos x\right)\left(-\sin x\right)$$

$$= -2 \sin x \cos x = -\sin 2x$$

$$\frac{d}{dx}\left(\cos^2\left(\frac{\pi}{3}+x\right)\right) = 2\cos\left(\frac{\pi}{3}+x\right)\left(-\sin\left(\frac{\pi}{3}+x\right)\right)$$

$$= -2\sin\left(\frac{\pi}{3}+x\right)\cos\left(\frac{\pi}{3}+x\right)$$

$$= -2\sin\left(\frac{\pi}{3}+x\right)\cos\left(\frac{\pi}{3}+x\right)$$
$$= -\sin\left(\frac{2\pi}{3}+2x\right)$$

equivalent

$p \Rightarrow q$ if p, then q (proposition)

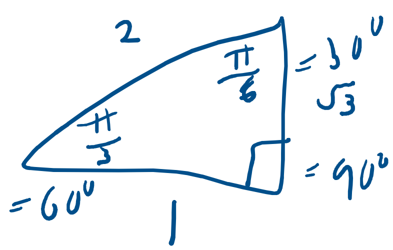
$q \Rightarrow p$ if q, then p (converse)

$\sim p \Rightarrow \sim q$ if not p, then not q (inverse)

$\sim q \Rightarrow \sim p$ if not q, then not p (contrapositive)

18. Show that $f(x) = \cos^2 x + \cos^2\left(\frac{\pi}{3}+x\right) - \cos x\cos\left(\frac{\pi}{3}+x\right)$ is a constant function. What is its value?

$$f(0) = \cos^2 0 + \cos^2\left(\frac{\pi}{3}\right) - (\cos 0)\left(\cos\frac{\pi}{3}\right)$$
$$= 1 + \frac{1}{4} - (1)\left(\frac{1}{2}\right)$$
$$= \frac{4+1-2}{4} = \frac{3}{4}$$



$$\frac{\pi}{3} + \frac{\pi}{6} = \frac{2\pi+\pi}{6} = \frac{3\pi}{6} = \frac{\pi}{2}$$

5.1
Memorize

An **antiderivative** $F(x)$ of a function $f(x)$ is a function whose derivative is $f(x)$. In other words, $F'(x) = f(x)$.

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Memorize

Suppose that $F(x)$ and $G(x)$ are antiderivatives of a function $f(x)$. Then $F(x)$ and $G(x)$ differ only by a constant. That is, $F(x) = G(x) + C$ for some constant C.

Memorize

To find *all* antiderivatives of a function, it is necessary only to find *one* antiderivative and then add a generic constant to it.

Memorize

The **indefinite integral** of a function $f(x)$ is denoted by

$$\int f(x)\,dx$$

and represents the entire family of antiderivatives of $f(x)$.

Memorize

$$\frac{d}{dx}\left(\int f(x)\,dx\right) = f(x)$$

$$\frac{dF}{dx} = f(x) = F'(x)$$

$$\Rightarrow dF = F'(x)\,dx$$

$$\Rightarrow dF = f(x)\,dx$$

Memorize

Power Formula: $$\int x^n\,dx = \begin{cases} \frac{x^{n+1}}{n+1} & \text{if } n \neq -1 \\ \ln|x| & \text{if } n = -1 \end{cases}$$

Memorize

Let f and g be functions and let k be a constant. Then:

1. $\int k\, f(x)\, dx = k \int f(x)\, dx$

2. $\int (f(x)+g(x))\, dx = \int f(x)\, dx + \int g(x)\, dx$

3. $\int (f(x)-g(x))\, dx = \int f(x)\, dx - \int g(x)\, dx$

Memorize

For any functions $f_1, \ldots, f_n$ and constants $k_1, \ldots, k_n$,

$$\int (k_1 f_1(x) + \cdots + k_n f_n(x))\, dx = k_1 \int f_1(x)\, dx + \cdots + k_n \int f_n(x)\, dx . \qquad (5.1)$$

i.e., we can integrate term by term

Memorize

$$\int \cos x\, dx = \sin x + C$$

$$\int \sin x\, dx = -\cos x + C$$

$$\int \sec^2 x\, dx = \tan x + C$$

$$\int \sec x \tan x\, dx = \sec x + C$$

$$\int \csc x \cot x\, dx = -\csc x + C$$

$$\int \csc^2 x\, dx = -\cot x + C$$

Memorize

$$\int e^x\, dx = e^x + C$$

Memorize or be able to derive

Free fall motion: At time $t \geq 0$:

acceleration:	$a(t) = -g$
velocity:	$v(t) = -gt + v_0$
position:	$s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0$
initial conditions:	$s_0 = s(0),\ v_0 = v(0)$

5.2

Memorize

The **definite integral** of a function $f(x)$ over an interval $[a, b]$ is denoted by

$$\int_a^b f(x)\, dx$$

and represents the sum of the infinitesimals $f(x)\, dx$ for all x in $[a, b]$.

Memorize

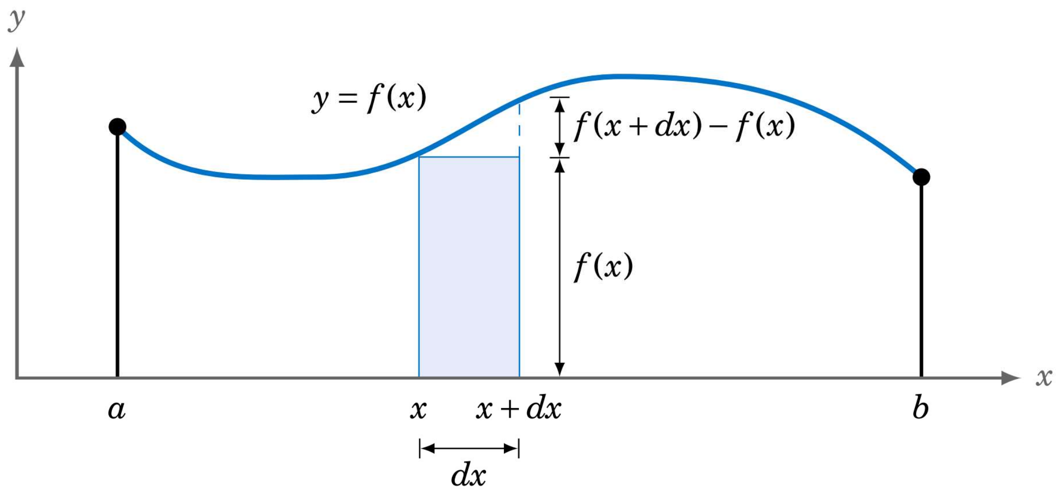


Figure 5.2.1 The infinitesimal $f(x)\, dx$ as the area of a rectangle

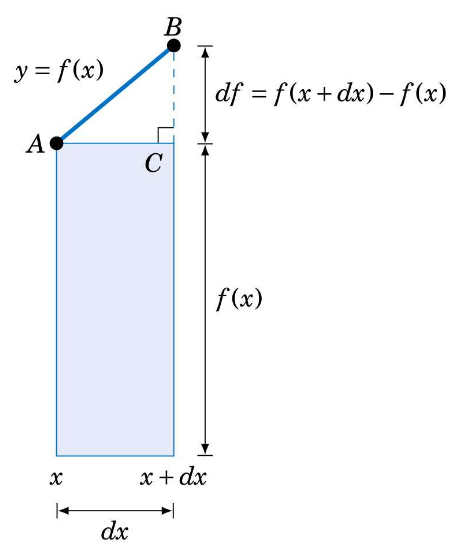


Figure 5.2.2 Area under the curve $y = f(x)$ over $[x, x + dx]$

$$\text{Area of } \triangle ABC = \frac{1}{2}(\text{base}) \times (\text{height}) = \frac{1}{2}(dx)(df) = \frac{1}{2}(dx)(f'(x)\, dx) = \frac{1}{2} f'(x)(dx)^2 = 0$$

Memorize

For a function $f(x) \geq 0$ over $[a, b]$, the **area under the curve** $y = f(x)$ between $x = a$ and $x = b$, denoted by A, is given by

$$A = \int_a^b f(x)\,dx$$

and represents the area of the region R bounded above by $y = f(x)$, bounded below by the x-axis, and bounded on the sides by $x = a$ and $x = b$ (with $a < b$).

Memorize

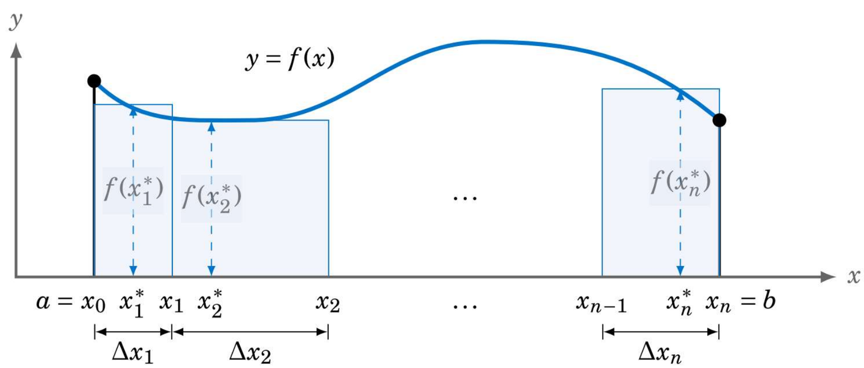
Let R be the region bounded by $y = f(x)$ and the x-axis between $x = a$ and $x = b$. If $f(x) \leq 0$ over $[a, b]$, then

$$\int_a^b f(x)\,dx = \text{the negative of the area of } R$$

If $f(x)$ changes sign over $[a, b]$, then

$$\int_a^b f(x)\,dx = \text{the net area of } R,$$

where the parts of R above the x-axis count as positive area and the parts below count as negative area.



$$\text{Area } A = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

We let n approach infinity and the largest Δx_i approaches zero.

Memorize

For real numbers $a_1, a_2, \ldots, a_n$ and an integer $n \geq 1$,

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

is the sum of $a_1, \ldots, a_n$. The symbol Σ is called the **summation sign**, which is the Greek capital letter Sigma.

$$\sum_{i=1}^{3} a_i = a_1 + a_2 + a_3$$

$$\sum_{j=1} a_j = a_1 + a_2 + a_3$$

$$\sum_{k=4}^{6} b_k = b_4 + b_5 + b_6$$

Memorize

Let $a_1, a_2, \ldots, a_n$, and $b_1, b_2, \ldots, b_n$ be real numbers, and let c be a constant. Then:

(1) $\displaystyle\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

(2) $\displaystyle\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

(3) $\displaystyle\sum_{k=1}^{n} c a_k = c\sum_{k=1}^{n} a_k$

(4) $\displaystyle\sum_{k=1}^{n} a_k = \sum_{i=1}^{n} a_i$ (i.e. the sum is independent of the summation index letter)

Supplied

Let $n \geq 1$ be a positive integer. Then:

(1) $\displaystyle\sum_{k=1}^{n} 1 = n$

(2) $\displaystyle\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

(3) $\displaystyle\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(4) $\displaystyle\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$

(5) $\displaystyle\sum_{k=1}^{n} k^4 = 1^4 + 2^4 + \cdots + n^4 = \frac{n(n+1)(6n^3+9n^2+n-1)}{30}$

Let $n \geq 1$ be a positive integer. Then:

(1) $\sum_{k=1}^{n} 1 = n$

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5.1

For Exercises 1-15, evaluate the given indefinite integral.

2. $\int 3\cos x \, dx = 3\int \cos x \, dx = \boxed{3\sin x + C}$

10. $\int \frac{1}{3\sqrt{x}} \, dx = \frac{1}{3}\int x^{-\frac{1}{2}} \, dx$

$= \frac{1}{3} \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = \left(\frac{2}{3}\right) x^{\frac{1}{2}} + C$

$= \frac{2x^{\frac{1}{2}}}{3} + C$

$= \frac{2}{3}\sqrt{x} + C$

Scientific Notebook

$\int_0^1 \sin(x)dx = 1 - \cos 1 \approx 0.45970$

$\sin(x)$ Approximate integral (left boxes) is $\frac{1}{10} \sum_{i_3=0}^{9} \sin \frac{1}{10} i_3 = 0.41724$

$\sin(x)$ Approximate integral (left boxes) is $\frac{1}{100} \sum_{i_4=0}^{99} \sin \frac{1}{100} i_4 = 0.45549$

$\sin(x)$ Approximate integral (right boxes) is $\frac{1}{10} \sum_{i_5=1}^{10} \sin \frac{1}{10} i_5 = 0.50139$

$\sin(x)$ Approximate integral (right boxes) is $\frac{1}{100} \sum_{i_6=1}^{100} \sin \frac{1}{100} i_6 = 0.4639$