04-01-26 MTH 263

4.2 Curve Sketching
page 118: 1, 3, 9

4.4 The Mean Value Theorem
page 132: 1, 5, 7, 18

4.2: 3

For Exercises 1-8 sketch the graph of the given function. Find all local maxima and minima, inflection points, where the function is increasing or decreasing, where the function is concave up or concave down, and indicate any asymptotes.

**3.** $f(x) = xe^{-x}$

$$f'(x) = x\frac{d}{dx}(e^{-x}) + e^{-x}\frac{dx}{dx}$$

$$= x(-e^{-x}) + e^{-x}(1)$$

$$f'(x) = -xe^{-x} + e^{-x}$$

Let $u = -x$

$$\frac{d}{dx} e^{-x} = \frac{d}{dx} e^{u(x)} = e^{u(x)}\frac{du}{dx} = e^{-x}(-1)$$

$$\boxed{f'(x) = e^{-x}(1-x)}$$ defined for all $x$

$e^{-x}(1-x) = 0$ solve for $x$

$\Rightarrow e^{-x} = 0$ or $1 - x = 0$

but $e^{-x} > 0$ for all $x$ | $\boxed{x = 1}$ critical point

$$f''(x) = \frac{d}{dx}\left(e^{-x}(1-x)\right)$$

$$= e^{-x}\frac{d}{dx}(1-x) + (1-x)\frac{d}{dx}(e^{-x})$$

$$= e^{-x}(-1) + (1-x)(-e^{-x})$$

$$= -e^{-x}(1 + 1 - x)$$

$$\boxed{f''(x) = -e^{-x}(2-x)}$$

finish on your own

4.2

**9.** Write $U/\epsilon$ and $C_V/k_B$ from Example 4.13 as functions of $x = \tau/\epsilon$. You do not need to sketch the graphs.

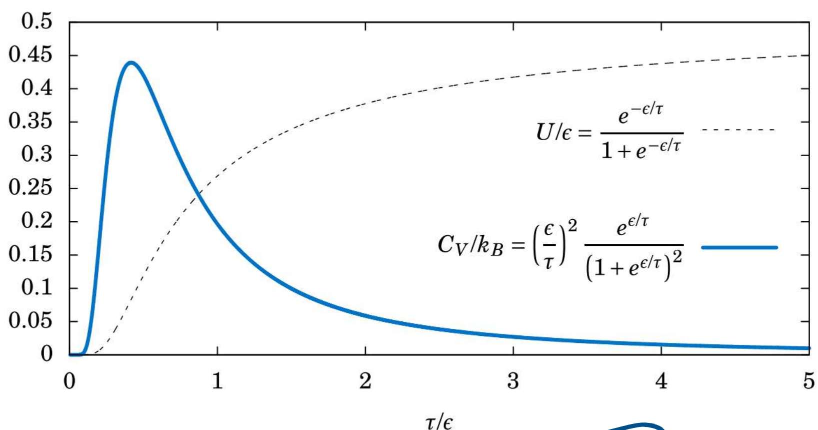
***Example 4.13***

For a single particle with two states—energy 0 and energy $\epsilon$—in thermal contact with a reservoir at temperature $\tau$, the average energy $U$ and heat capacity $C_V$ are given by

$$U = \epsilon \frac{e^{-\epsilon/\tau}}{1+e^{-\epsilon/\tau}} \quad \text{and} \quad C_V = k_B \left(\frac{\epsilon}{\tau}\right)^2 \frac{e^{\epsilon/\tau}}{\left(1+e^{\epsilon/\tau}\right)^2}$$

where $k_B \approx 1.38065 \times 10^{-23}$ J/K is the *Boltzmann constant*. The graph below shows both quantities as functions of $\tau/\epsilon$ (*not* $\epsilon/\tau$, as you might expect). See Exercise 9.

Average Energy $U/\epsilon$ vs Heat Capacity $C_V/k_B$




Let $x = \frac{\tau}{\varepsilon} \Rightarrow \boxed{\frac{\varepsilon}{\tau} = \frac{1}{x}} \Rightarrow \boxed{\frac{-\varepsilon}{\tau} = \frac{-1}{x}}$

$$\boxed{\frac{U}{\varepsilon} = \frac{e^{-\frac{1}{x}}}{1+e^{-\frac{1}{x}}}} \qquad \frac{C_V}{k_B} = ?$$

4.4
Supplied

**Mean Value Theorem:** Let $a$ and $b$ be real numbers such that $a < b$, and suppose that $f$ is a function such that
**(a)** $f$ is continuous on $[a, b]$, and
**(b)** $f$ is differentiable on $(a, b)$

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**(a)** $f$ is continuous on $[a,b]$, and
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Then there is at least one number $c$ in the interval $(a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \quad (4.3)$$

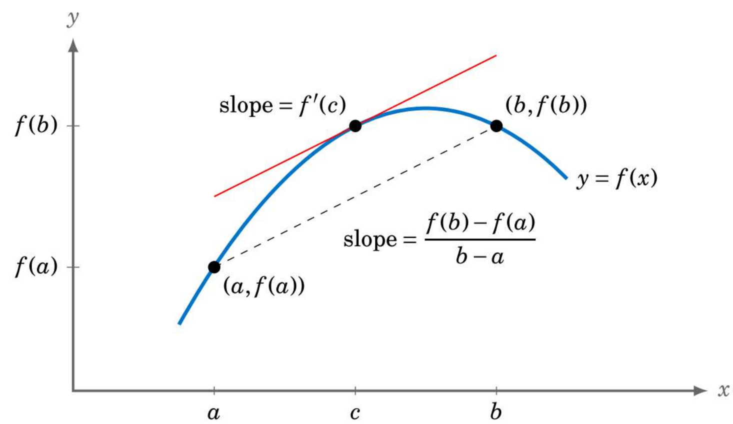



**Figure 4.4.1** Mean Value Theorem: parallel tangent line and secant line

supplied

**Rolle's Theorem:** Let $a$ and $b$ be real numbers such that $a < b$, and suppose that $f$ is a function such that
**(a)** $f$ is continuous on $[a,b]$,
**(b)** $f$ is differentiable on $(a,b)$, and
**(c)** $f(a) = f(b) = 0$.
Then there is at least one number $c$ in the interval $(a,b)$ such that $f'(c) = 0$.

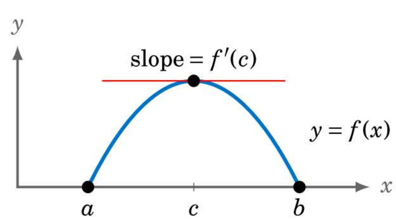



**Figure 4.4.2**

Definition: a corollary is a direct consequence of some theorem.

Definition: a lemma is a theorem which is preliminary to a major theorem.

Memorize

If $f$ is a differentiable function on an interval $I$ such that $f'(x) = 0$ for all $x$ in $I$, then $f$ is a constant function on $I$.

Memorize

Let $f$ be a differentiable function on an interval $I$. Then:

**(a)** If $f' > 0$ on $I$ then $f$ is increasing on $I$.

**(b)** If $f' < 0$ on $I$ then $f$ is decreasing on $I$.

Not on final exam.

**Mean Value Theorem (alternative form):** Let $a$ and $h > 0$ be real numbers, and suppose that $f$ is a function such that

**(a)** $f$ is continuous on $[a, a+h]$, and

**(b)** $f$ is differentiable on $(a, a+h)$.

Then there is a number $\theta$ in the interval $(0,1)$ such that

$$f(a+h) - f(a) = h f'(a+\theta h) . \tag{4.4}$$

**Extended Mean Value Theorem:** Let $a$ and $b$ be real numbers such that $a < b$, and suppose that $f$ and $g$ are functions such that

**(a)** $f$ and $g$ are continuous on $[a,b]$,

**(b)** $f$ and $g$ are differentiable on $(a,b)$, and

**(c)** $g'(x) \neq 0$ for all $x$ in $(a,b)$.

Then there is at least one number $c$ in the interval $(a,b)$ such that

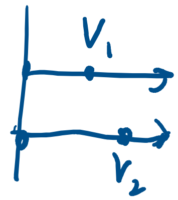
$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} . \tag{4.5}$$

Supplied

**Darboux's Theorem:** If $f$ is a differentiable function on a closed interval $[a,b]$ then its derivative $f'$ attains every value between $f'(a)$ and $f'(b)$.

4.4:

**2.** Suppose that two horses run a race starting together and ending in a tie. Show that, at some time during the race, they must have had the same speed.




Prove : At some time $t$ $V_1(t) = V_2(t)$

$0 < t <$ ending time

use Rolle's Thm.

finish later

Your name MTH 263 bonus quiz 2 calculator OK.

1. Write the formal definition of $\lim_{x \to c} f(x) = L$.

$\forall \varepsilon > 0, \exists \delta > c$ such that

$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$

2. Use calculus to find the global max and global min of $f(x) = 3x + 1$ on the interval $[1,5]$.

$f'(x) = 3$ defined for all $x$

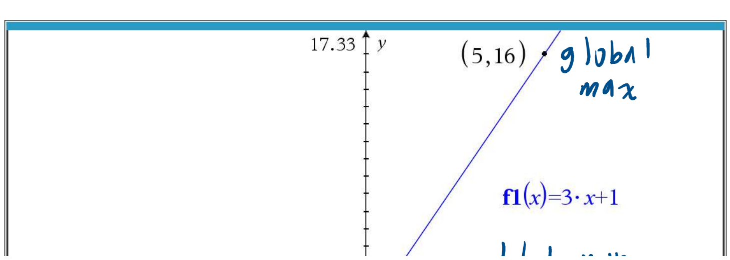
$\forall x \; f'(x) \neq 0$

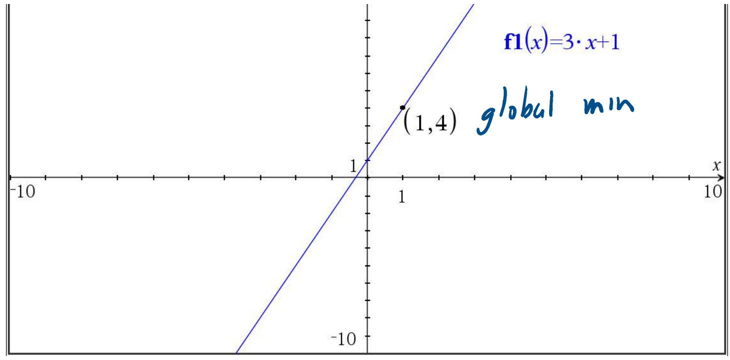
$\therefore$ No critical point on $(1,5)$

check endpoints

$f(1) = 3(1) + 1 = 4$ global min

$f(5) = 3(5) + 1 = 16$ global max




f1(x)=3·x+1
(1,4) global min
1
-10
1
10
x
-10