03-30-26 MTH 263

4 Applications of Derivatives

4.1 Optimization

page 108: 1, 3, 7, 13, 17

4.2 Curve Sketching

page 118: 1, 3, 9

Before class notes

?

4.1

13. For an inventory model with a constant order quantity $Q > 0$ and a constant linear inventory depletion rate D, the *total unit cost* TC to maintain an average inventory of $Q/2$ units is

$$TC = C + \frac{P}{Q} + \frac{(I+W)Q}{2D}$$

not needed

where C is the capital investment cost, P is the cost per order, I is the per unit interest charge per unit time, and W is the overall inventory holding cost. Find the value of Q that minimizes TC.

Assume $Q \neq$ constant

Q = independent variable

assume C, P, I, W, D are positive constants

$$\frac{d}{dQ}(TC) = \frac{d}{dQ}(C) + \frac{d}{dQ}(PQ^{-1}) + \frac{d}{dQ}\left(\frac{(I+W)Q}{2D}\right)$$

$$= 0 - PQ^{-2} + \frac{I+W}{2D}$$

$$\boxed{\frac{d}{dQ}(TC) = -\frac{P}{Q^2} + \frac{I+W}{2D}}$$

$$-\frac{P}{Q^2} + \frac{I+W}{2D} = 0$$

$$\frac{P}{Q^2} = \frac{I+W}{2D}$$

$$Q^2(I+W) = 2PD$$

$$Q^2(I+W) = 2PD$$

$$Q^2 = \frac{2PD}{I+W}$$

$$\boxed{Q = \sqrt{\frac{2PD}{I+W}}}$$

2nd deriv test

$$\frac{d}{dQ}\left(-PQ^{-2} + \frac{I+W}{2D}\right)$$

$$= 2PQ^{-3} + 0$$

$$\frac{d^2(TC)}{dQ^2} = \frac{2P}{Q^3} > 0$$

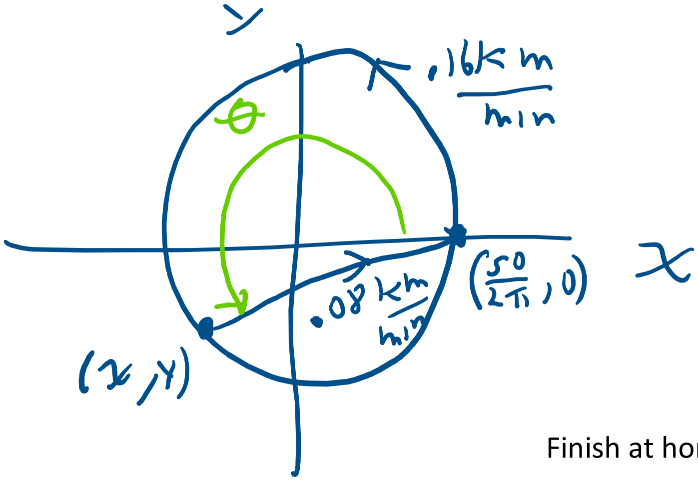
$\Rightarrow$ local min

no other critical point on open interval

$\therefore$ global min

4.1: 17

17. A certain jogger can run 0.16 km/min, and walks at half that speed. If he runs along a circular trail with circumference 50 km and then—before completing one full circle—walks back straight across to his starting point, what is the maximum time he can spend on the run/walk?



r = radius

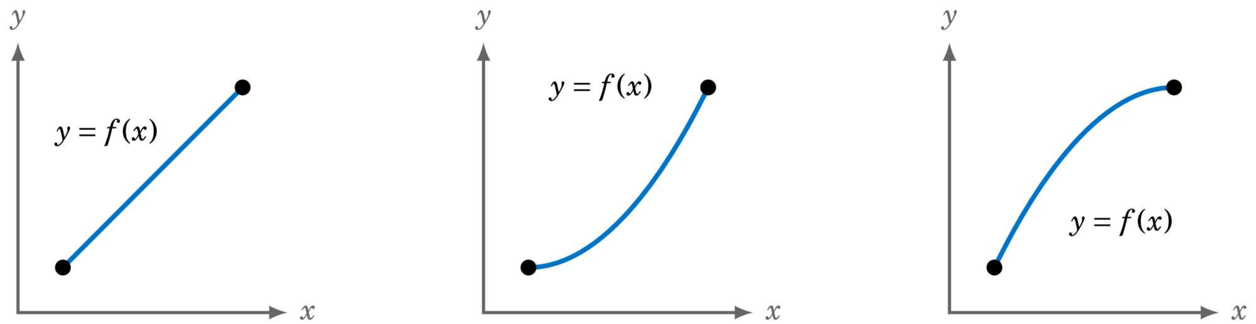
$$\boxed{r = \frac{50\text{ km}}{2\pi}}$$

$0 \le \theta < 2\pi$

θ = angle run along circle

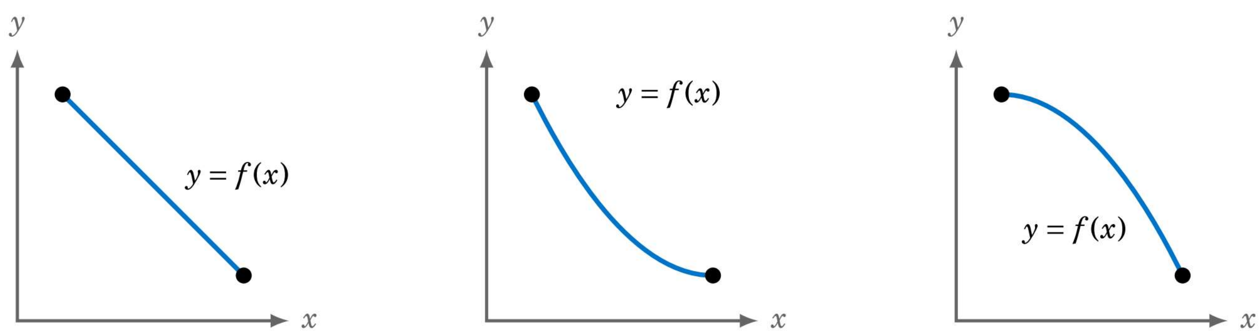
Finish at home.

4.2
Memorize



(a) $f'' = 0$: straight **(b)** $f'' > 0$: concave up **(c)** $f'' < 0$: concave down

Figure 4.2.1 Increasing function f: $f' > 0$, different signs for f''



(a) $f'' = 0$ **(b)** $f'' > 0$: concave up **(c)** $f'' < 0$: concave down

Figure 4.2.2 Decreasing function f: $f' < 0$, different signs for f''

Memorize

Concavity Theorem: Suppose that f is a twice-differentiable function on $[a, b]$. Then:

(a) If $f''(x) > 0$ on (a, b) then $f(x)$ is below the line $l(x)$ joining the points $(a, f(a))$ and $(b, f(b))$ for all x in (a, b).

(b) If $f''(x) < 0$ on (a, b) then $f(x)$ is above the line $l(x)$ joining the points $(a, f(a))$ and $(b, f(b))$ for all x in (a, b).

Memorize

A function f has an **inflection point** at $x = c$ if the concavity of f changes around $x = c$. That is, the function goes from concave up to concave down, or vice versa.

Memorize

First Derivative Test: For a continuous function f on an interval I, let $x = c$ be a number in I such that $f(c)$ is defined, and either $f'(c) = 0$ or $f'(c)$ does not exist. Then:

(a) If $f'(x)$ changes from negative to positive around $x = c$ then f has a local minimum at $x = c$.

(b) If $f'(x)$ changes from positive to negative around $x = c$ then f has a local maximum at $x = c$.

Supplied

N^th Derivative Test: A function f with continuous derivatives of all orders up to and including $n > 1$ at $x = c$ has either a local minimum, local maximum or inflection point at $x = c$ if and only if

$$f^{(k)}(c) = 0 \text{ for } k = 1, 2, \ldots, n-1 \quad \text{and} \quad f^{(n)}(c) \neq 0$$

(i.e. the n^{th} derivative is the first nonzero derivative at $x = c$). If so, then:

(a) If $n > 1$ is even and $f^{(n)}(c) > 0$ then f has a local minimum at $x = c$.

(b) If $n > 1$ is even and $f^{(n)}(c) < 0$ then f has a local maximum at $x = c$.

(c) If $n > 1$ is odd then f has an inflection point at $x = c$.

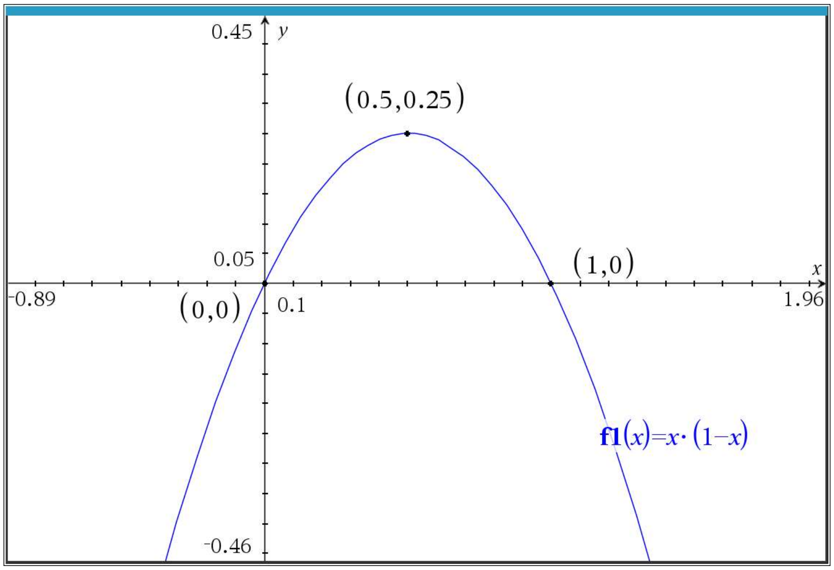
Your Name MTH 263 quiz 4 Write each problem.

1. Use calculus to prove the following.

Prove that for $0 \le p \le 1$, $p(1-p) \le \frac{1}{4}$.

Let $f(p) = p(1-p)$

Find global max of $f(p)$ on $[0, 1]$



$f(p) = p - p^2$

$f'(p) = 1 - 2p = 0$

$1 = 2p$

$2p = 1$

$p = \frac{1}{2}$ critical point

$f''(p) = -2 < 0$

$\therefore$ local max → also global max

$f(0) = 0$ global min

$f(1) = 0$ global min

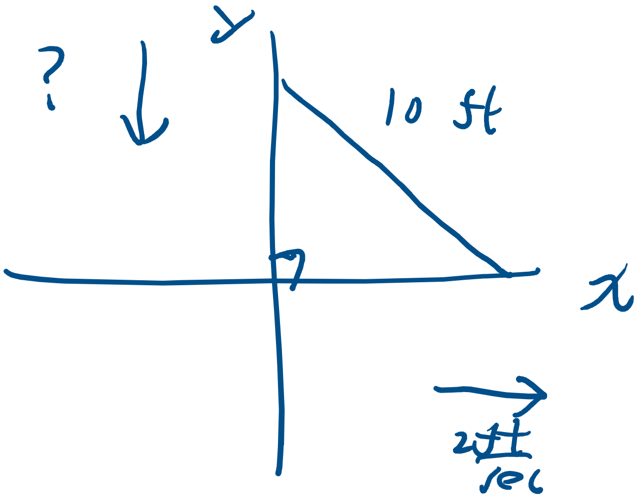
$f(\frac{1}{2}) = (\frac{1}{2})(1 - \frac{1}{2}) = (\frac{1}{2})^2 = \frac{1}{4}$

$\therefore f(p) \leq \frac{1}{4}$ on $[0, 1]$

2.

A **10-foot ladder** is leaning against a vertical wall. The bottom of the ladder begins to slide away from the wall at a constant rate of **2 feet per second**.

How fast is the top of the ladder sliding down the wall at the moment when the bottom of the ladder is 6 feet from the wall?



Let $x(t)$ = position (ft) of bottom of ladder at time t (sec)

Let $y(t)$ = position (ft) of top of ladder at time t

Find $\frac{dy}{dt}$ when $x = 6$ ft

$\frac{dx}{dt} = 2 \frac{ft}{sec}$

$$x^2 + y^2 = 10^2 = 100$$

$$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(100)$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

$$y\frac{dy}{dt} = -x\frac{dx}{dt}$$

$$\boxed{\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}}$$

$$\frac{dy}{dt} = \left(-\frac{6\text{ ft}}{y}\right)\left(2\frac{\text{ft}}{\text{sec}}\right)$$

$$x^2 + y^2 = 100\text{ ft}^2$$

$$(6\text{ ft})^2 + y^2 = 100\text{ ft}^2$$

$$y^2 = (100 - 36)\text{ ft}^2$$

$$y^2 = 64\text{ ft}^2$$

$$\boxed{y = 8\text{ ft}}$$

$$\frac{dy}{dt} = \left(-\frac{\overset{3}{\cancel{6}}\ \cancel{\text{ft}}}{\underset{4}{\cancel{8}}\ \cancel{\text{ft}}}\right)\left(2\frac{\text{ft}}{\text{sec}}\right)$$

$$\boxed{\frac{dy}{dt} = \left(-\frac{3}{2}\right)\frac{\text{ft}}{\text{sec}}}$$

4.2:

For Exercises 1-8 sketch the graph of the given function. Find all local maxima and minima, inflection points, where the function is increasing or decreasing, where the function is concave up or concave down, and indicate any asymptotes.

2. $f(x) = x^3 - 3x^2 + 1$

$f'(x) = 3x^2 - 6x$ defined all x

$3x^2 - 6x = 0 \Rightarrow x^2 - 2x = 0 \Rightarrow x(x-2) = 0$

$x = 0, 2$

$\Rightarrow x = 0$ critical point

$f''(2) = 12 - 6 = 6 > 0$

local min

$f''(x) = 6x - 6$

$f''(0) = -6 < 0$

$\therefore$ local max at $x = 0$

$f(0) = 1$

$f(2) = 8 - 12 + 1$

$f(2) = -3$

$f''(x) = 0 = 6x - 6$

$\Rightarrow 6x = 6$

$\Rightarrow x = 1$

$f''(x) = 6x - 6 > 0$

$6x > 6$

$x > 1$ concave up $(1, \infty)$

$f''(x) = 6x - 6 < 0$ concave down $(-\infty, 1)$

$x < 1$

$\therefore$ inflection point $x = 1$

$f(1) = -1$