

3.3 Continuity

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3.4 Implicit Differentiation

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Exam 2, Wednesday, 03/11/26

2.4, 3.1 - 3.4

After class notes

3.3: 15

For Exercises 1-18, indicate whether the given function $f(x)$ is continuous or discontinuous at the given value $x = a$ by comparing $f(a)$ with $\lim_{x \rightarrow a} f(x)$.

$$15. f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0; \end{cases}$$

at $x = 0$

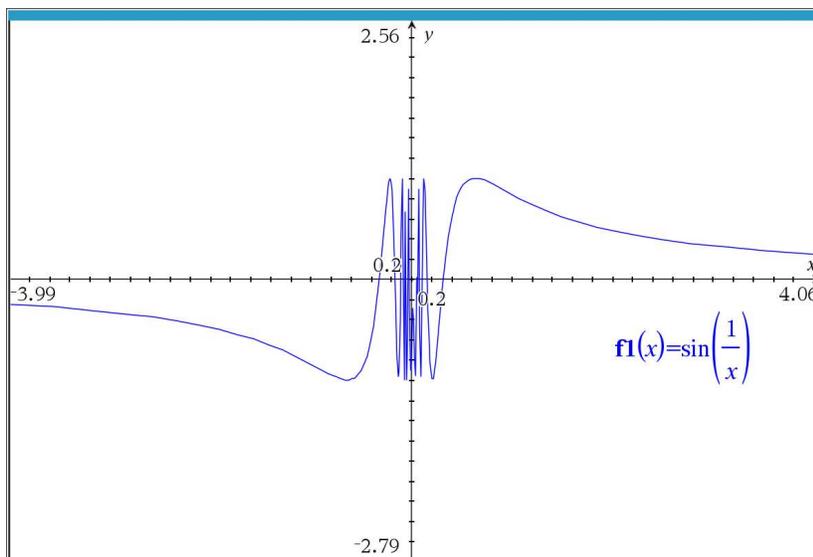
$$f(0) = 0$$

$$\lim_{x \rightarrow 0} f(x) = ?$$

A function f is **continuous** at $x = a$ if

$$\lim_{x \rightarrow a} f(x) = f(a). \quad (3.4)$$

A function is continuous on an interval I if it is continuous at every point in the interval. For a closed interval $I = [a, b]$, a function f is continuous on I if it is continuous on the open interval (a, b) and if $\lim_{x \rightarrow a^+} f(x) = f(a)$ (i.e. f is **right continuous** at $x = a$) and $\lim_{x \rightarrow b^-} f(x) = f(b)$ (i.e. f is **left continuous** at $x = b$). A function is **discontinuous** at a point if it is not continuous there. A continuous function is one that is continuous over its entire domain.



As x approaches 0, $\sin(1/x)$ oscillates faster and faster between -1 and $+1$, never staying near 0. Therefore, the function is discontinuous at 0.

3.3: 5

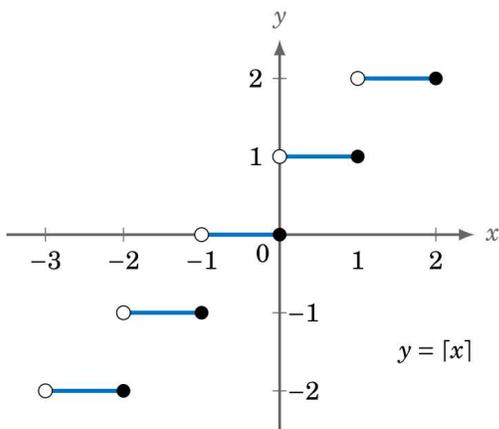
For Exercises 1-18, indicate whether the given function $f(x)$ is continuous or discontinuous at the given value $x = a$ by comparing $f(a)$ with $\lim_{x \rightarrow a} f(x)$.

5. $f(x) = \lceil x \rceil$; at $x = 0$

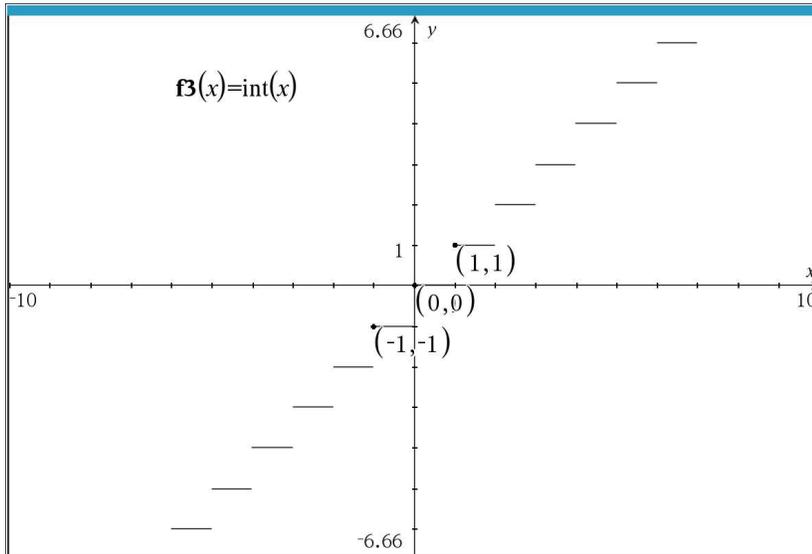
supplied

Similarly, the *ceiling function* $\lceil x \rceil$ is defined as

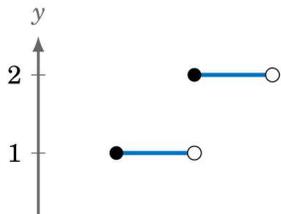
$\lceil x \rceil =$ the smallest integer greater than or equal to x .

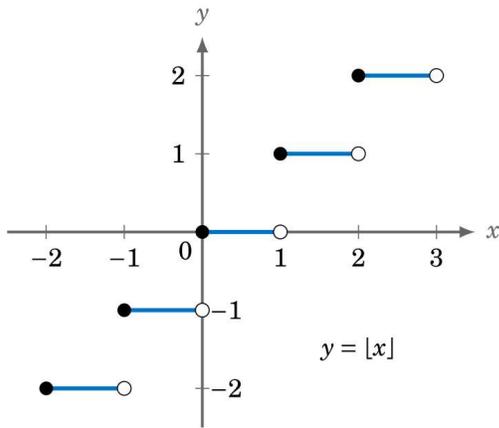


(b) Ceiling function $\lceil x \rceil$



$\int \int \text{int}(x) = \lfloor x \rfloor$





(a) Floor function $[x]$

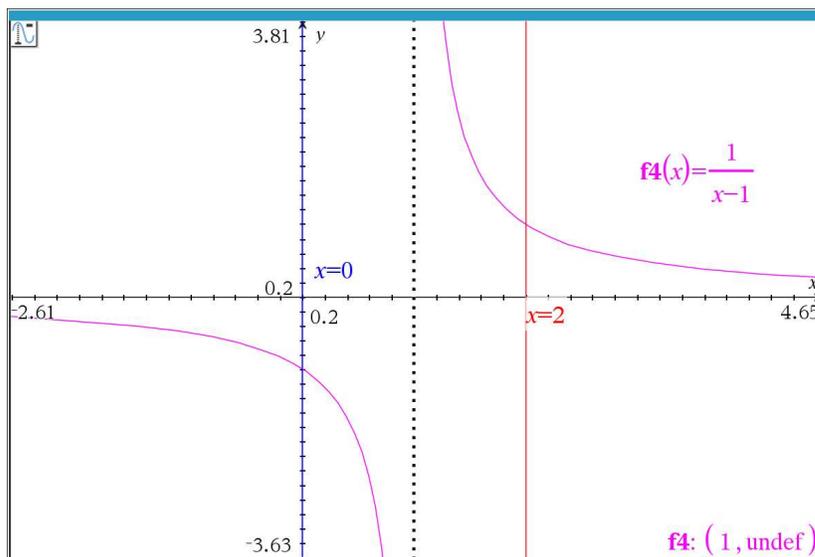
$$\text{Int}(\text{int}(x)) = \lfloor x \rfloor$$

3.3: 29

29. Let $f(x) = 1/(x-1)$. Then $f(0) = -1 < 0$ and $f(2) = 1 > 0$. Can you conclude by the Intermediate Value Theorem that $f(x)$ must be 0 for some x in $[0, 2]$? Explain.

Intermediate Value Theorem: If f is a continuous function on a closed interval $[a, b]$ then f attains every value between $f(a)$ and $f(b)$.

$f(1)$ is not defined $0 < 1 < 2$
 $\therefore f$ is not continuous on $[0, 2]$
 \therefore IVT does not apply



3.3: 25

25. Show that there is a number x such that $x^5 - x = 3$.

$$\text{Let } f(x) = x^5 - x - 3$$

$\forall x, f(x)$ is continuous

$$\text{Let } f(x) = x^5 - x - 3$$

$\forall x, f(x)$ is continuous

$$\exists x \text{ such } x^5 - x = 3$$

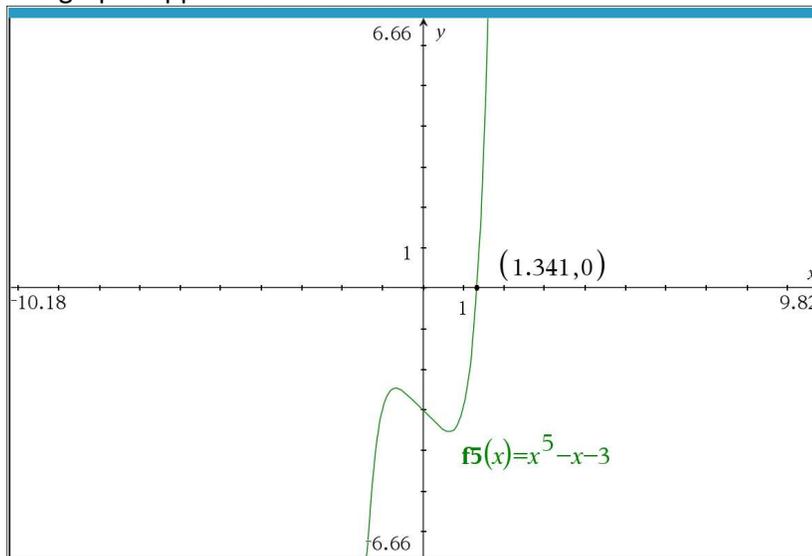
$$\Leftrightarrow \exists x \ f(x) = 0$$

$$f(1) = 1^5 - 1 - 3 = -3 < 0$$

$$f(2) = 2^5 - 2 - 3 = 32 - 5 = 27 > 0$$

$$\therefore \text{IVT} \Rightarrow \exists 0 \text{ of } f \text{ in } (0, 2)$$

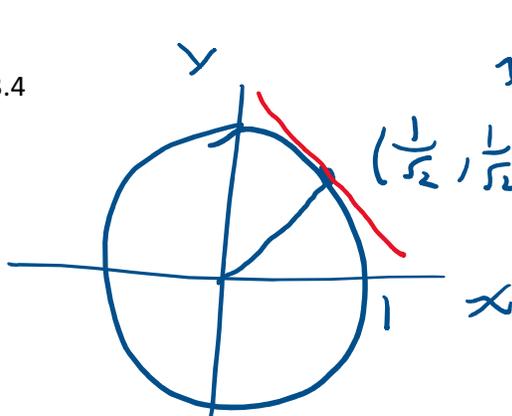
The graph supports our conclusion.



$x \approx 1.341$ is a zero of f

and $0 < 1.341 < 2$

3.4



Find $\frac{dy}{dx}$ at $x = \frac{1}{\sqrt{2}}$

2 2 1

$$x^2 + y^2 = 1$$

solve for y

$$y^2 = 1 - x^2$$

$$y = \pm \sqrt{1 - x^2}$$

$$y_1 = \sqrt{1 - x^2}, \quad y_2 = -\sqrt{1 - x^2}$$

$$y_1 = (1 - x^2)^{\frac{1}{2}}$$

$$\frac{dy_1}{dx} = \left(\frac{1}{2}\right)(1 - x^2)^{-\frac{1}{2}}(-2x)$$

$$\frac{dy_1}{dx} = \frac{-x}{\sqrt{1 - x^2}}$$

$$\frac{dy_1}{dx} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) = \frac{-\frac{1}{\sqrt{2}}}{\sqrt{1 - \left(\frac{1}{\sqrt{2}}\right)^2}}$$

$$= \frac{-\frac{1}{\sqrt{2}}}{\sqrt{1 - \frac{1}{2}}} = \frac{-\frac{1}{\sqrt{2}}}{\sqrt{\frac{1}{2}}}$$

$$= \frac{-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \boxed{-1}$$

Implicit diff.

$$x^2 + y^2 = 1$$

$$\text{Find } \frac{dy}{dx} \text{ at } \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(1)$$

$$2x + 2y \frac{dy}{dx} = 0$$

$$x + y \frac{dy}{dx} = 0$$

$$y \frac{dy}{dx} = -x$$

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

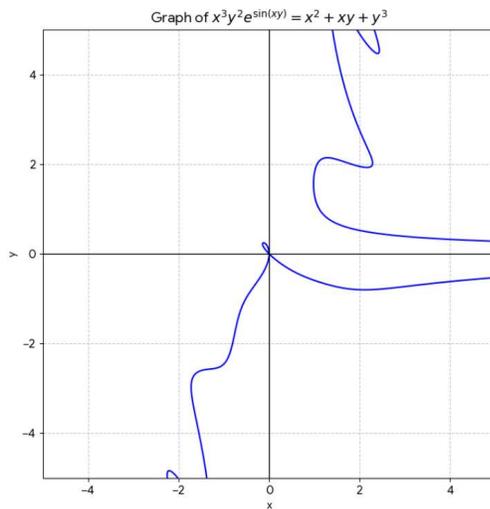
$$\frac{dy}{dx} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \boxed{-1}$$

3.4

For Exercises 1-9, use implicit differentiation to find $\frac{dy}{dx}$.

9. $x^3 y^2 e^{\sin(xy)} = x^2 + xy + y^3$

Google Gemini Graph



$$x^3y^2e^{\sin(xy)} = x^2 + xy + y^3$$

Please graph the relation. Then, find dy/dx

Analysis ▾

To find the derivative of this relation, we need to use **implicit differentiation**. Because y is not isolated, we treat it as a function of x and apply the product rule and chain rule across both sides.

1. The Relation and its Graph

The equation provided is:

$$x^3y^2e^{\sin(xy)} = x^2 + xy + y^3$$

This is a complex implicit relation. While it cannot be easily solved for y in terms of elementary functions, its graph reveals a curve where the transcendental and polynomial terms balance out.

2. Finding $\frac{dy}{dx}$ Step-by-Step

We differentiate both sides with respect to x :

$$\frac{d}{dx} [x^3y^2e^{\sin(xy)}] = \frac{d}{dx} [x^2 + xy + y^3]$$

Left Side (Product & Chain Rule)

The left side is a product of three terms: x^3 , y^2 , and $e^{\sin(xy)}$. Using the product rule $\frac{d}{dx}(uvw) = u'vw + uv'w + uvw'$:

1. Differentiate x^3 : $(3x^2) \cdot y^2e^{\sin(xy)}$
2. Differentiate y^2 : $x^3 \cdot (2y \frac{dy}{dx}) \cdot e^{\sin(xy)}$
3. Differentiate $e^{\sin(xy)}$: $x^3y^2 \cdot e^{\sin(xy)} \cos(xy) (y + x \frac{dy}{dx})$

Right Side

$$\frac{d}{dx} (x^2 + xy + y^3) = 2x + \left(y + x \frac{dy}{dx} \right) + 3y^2 \frac{dy}{dx}$$

3. Combining and Solving for $\frac{dy}{dx}$

Now we set the sides equal and group all terms containing $\frac{dy}{dx}$ on one side:

$$3x^2y^2e^{\sin(xy)} + 2x^3ye^{\sin(xy)}\frac{dy}{dx} + x^3y^2e^{\sin(xy)}\cos(xy)\left(y + x\frac{dy}{dx}\right) = 2x + y + x\frac{dy}{dx} + 3y^2$$

Distributing the $x^3y^2e^{\sin(xy)}\cos(xy)$ term:

$$3x^2y^2e^{\sin(xy)} + 2x^3ye^{\sin(xy)}\frac{dy}{dx} + x^3y^3e^{\sin(xy)}\cos(xy) + x^4y^2e^{\sin(xy)}\cos(xy)\frac{dy}{dx} = 2x + y + x\frac{dy}{dx} + 3y^2$$

Collecting $\frac{dy}{dx}$ terms:

$$\frac{dy}{dx} \left[2x^3ye^{\sin(xy)} + x^4y^2e^{\sin(xy)}\cos(xy) - x - 3y^2 \right] = 2x + y - 3x^2y^2e^{\sin(xy)} - x^3y^3e^{\sin(xy)}$$

Final Result:

$$\frac{dy}{dx} = \frac{2x + y - 3x^2y^2e^{\sin(xy)} - x^3y^3e^{\sin(xy)}\cos(xy)}{2x^3ye^{\sin(xy)} + x^4y^2e^{\sin(xy)}\cos(xy) - x - 3y^2}$$

3.4: 2

2. $xy = (x+y)^3$

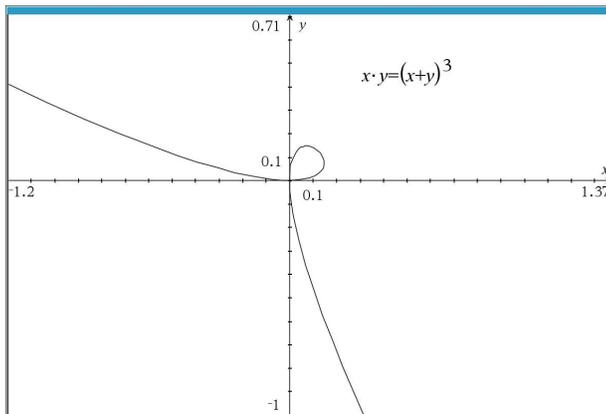
$x=0 \Rightarrow (0)(y) = (0+y)^3$

$\Rightarrow 0 = y^3$

$\Rightarrow y=0$

$\therefore (0,0)$ is a point on the graph

Find $\frac{dy}{dx}(0,0)$



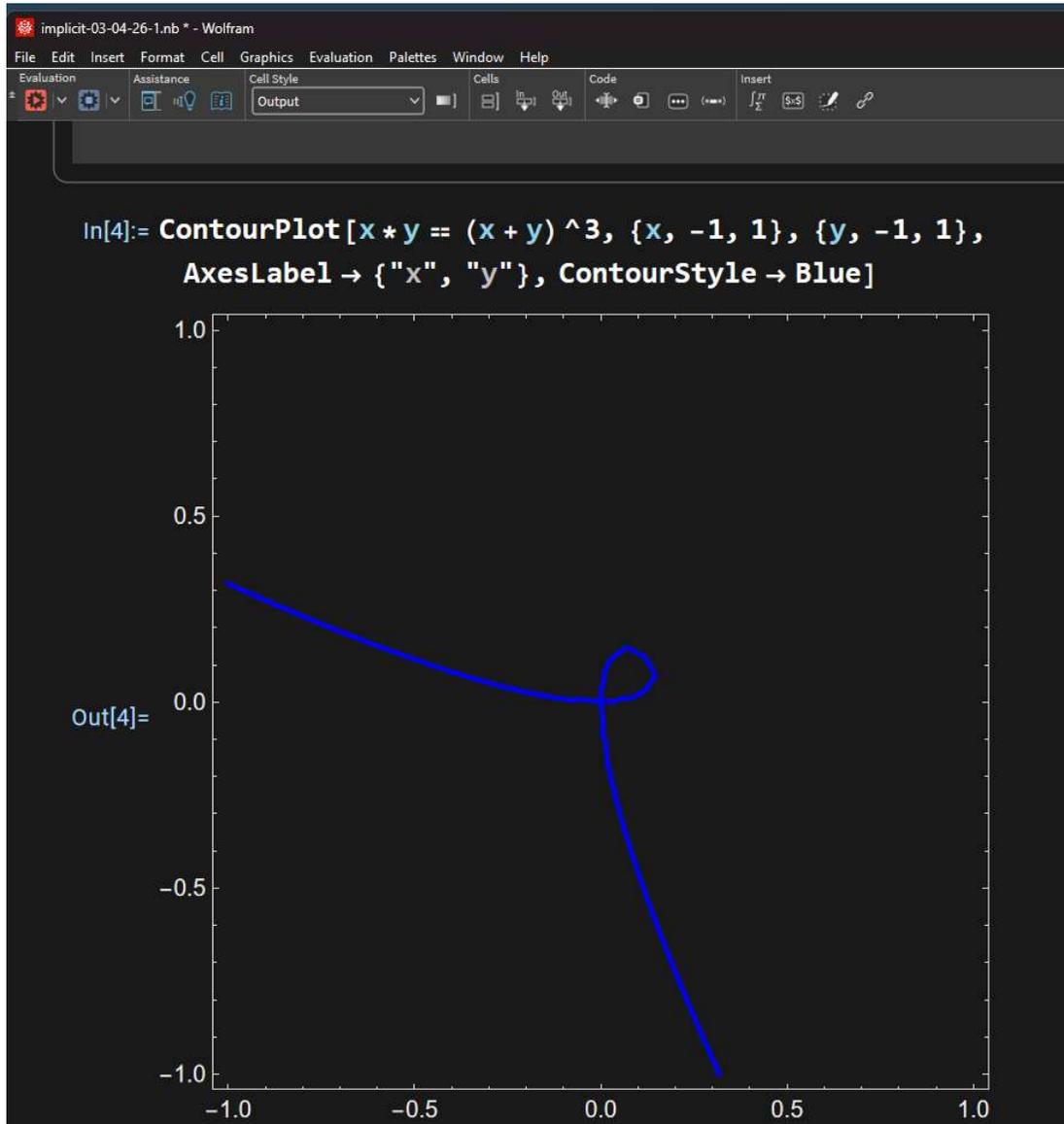
$$xy = (x+y)^3$$
$$\frac{d}{dx}(xy) = \frac{d}{dx}(x+y)^3$$

$$x\frac{dy}{dx} + y = 3(x+y)^2\left(1 + \frac{dy}{dx}\right)$$

$$x\frac{dy}{dx} - 3(x+y)^2\frac{dy}{dx} = 3(x+y)^2 - y$$

$$x \frac{dy}{dx} - 3(x+y) \frac{dy}{dx} = 3(x+y) - y$$

$$\frac{dy}{dx} = \frac{3(x+y) - y}{x - 3(x+y)}$$



What is the derivative at (0,0)? The graph intersects itself here.

Used Wolfram Language Evaluator -

Raw Interpreted

INPUT

code

$$\frac{dy}{dx} = \frac{3(x+y)^2 - y}{x-3}$$

$\frac{dy}{dx} /. \{x \rightarrow 0, y \rightarrow 0\}$

Copy Insert and evaluate

OUTPUT

Indeterminate

The derivative at the point (0, 0) is indeterminate because the denominator and numerator both evaluate to zero. This often occurs in cases where the behavior of the function is more complex, such as at points where the graph intersects itself or has a cusp.

In such cases, analyzing the behavior around the point or using a limit approach could provide more insight into the behavior of the graph at that point. Would you like to explore this further?

Yes. It appears that going from left to right towards (0,0) the derivative should approach zero, while looping around and approaching (0,0) from above would result in a vertical tangent line; hence the derivative would be undefined. Is this correct

Yes, your analysis is correct. The behavior of the curve near the point $(0, 0)$ can lead to different tangent line slopes depending on the direction from which you approach the point.

1. Approaching from the left or right: If you approach $(0, 0)$ horizontally (i.e., along paths where y remains constant), the derivative will approach zero. This suggests a horizontal tangent line.

2. Approaching from above or below: If you approach $(0, 0)$ vertically (i.e., along paths where x remains constant), the derivative would become undefined, implying a vertical tangent line.

These contrasting behaviors at $(0, 0)$ indicate that the graph has a more complex structure at this intersection point, such as a cusp or a self-intersection, where different sections of the curve meet with different slopes. Analyzing the limit of the derivative as you approach $(0, 0)$ from different directions can provide a more precise understanding of the curve's behavior at this point.

What limit calculation would support my conclusions? 



To support your conclusions about the behavior of the derivative at the point $(0, 0)$, you can calculate the limit of the derivative $\frac{dy}{dx}$ as you approach $(0, 0)$ from different paths:

- 1. Horizontal Approach:** Consider approaching $(0, 0)$ along the line $y = 0$.
- 2. Vertical Approach:** Consider approaching $(0, 0)$ along the line $x = 0$.

For the horizontal approach, we substitute $y = 0$ into the derivative and find the limit as $x \rightarrow 0$:

$$\lim_{x \rightarrow 0} \frac{3(x+0)^2 - 0}{x - 3(x+0)^2}$$

For the vertical approach, we substitute $x = 0$ into the derivative and find the limit as $y \rightarrow 0$:

$$\lim_{y \rightarrow 0} \frac{3(0+y)^2 - y}{0 - 3(0+y)^2}$$

Let's compute these limits to see the behavior of the derivative:

```
In[5]:= horizontalLimit = Limit[(3 (x + 0) ^ 2 - 0) / (x - 3 (x + 0) ^ 2), x -> 0];
verticalLimit = Limit[(3 (0 + y) ^ 2 - y) / (0 - 3 (0 + y) ^ 2), y -> 0];
{horizontalLimit, verticalLimit}
```

```
Out[8]= {0, Indeterminate}
```

$$\lim_{x \rightarrow 0} \left(\frac{3(x+0)^2 - 0}{x - 3(x+0)^2} \right) = \lim_{x \rightarrow 0} \frac{3x^2}{x - 3x^2} = \lim_{x \rightarrow 0} (3x) = \boxed{0}$$

$$\lim_{y \rightarrow 0} \left(\frac{3(0+y)^2 - y}{0 - 3(0+y)^2} \right) = \lim_{y \rightarrow 0} \frac{3y^2 - y}{-3y^2}$$

$$= \lim_{y \rightarrow 0} \frac{3y - 1}{-3y} = \lim_{y \rightarrow 0} \left(-1 - \frac{1}{3y} \right) = \pm \infty \begin{matrix} \text{from} \\ \text{top} \\ \text{or} \\ \text{bottom} \end{matrix} \text{ (dne)}$$

