02/25/26 MTH 263

3.2 Limits: Formal Definition

page 82: 1, 3, 5, 7, 13

3.3 Continuity

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supplied

Squeeze Theorem: Suppose that for some functions f, g and h there is a number $x_0 \geq 0$ such that

$$g(x) \leq f(x) \leq h(x) \text{ for all } x > x_0$$

and that $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} h(x) = L$. Then $\lim_{x\to\infty} f(x) = L$.

Similarly, if $g(x) \leq f(x) \leq h(x)$ for all $x \neq a$ in some interval I containing a, and if $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} f(x) = L$.

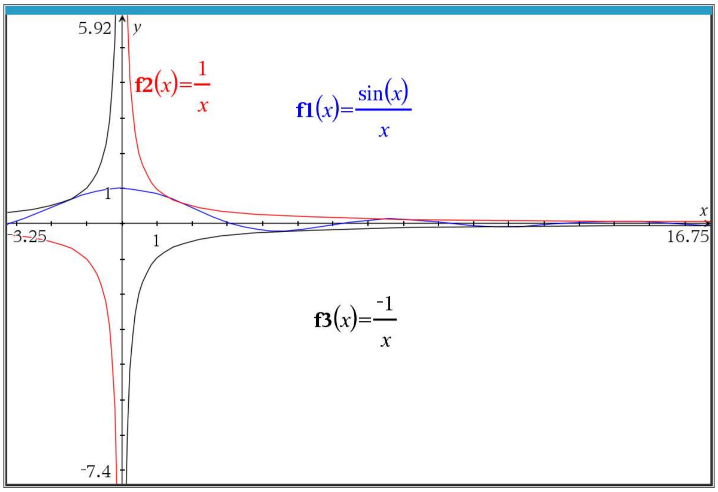
Example 3.21

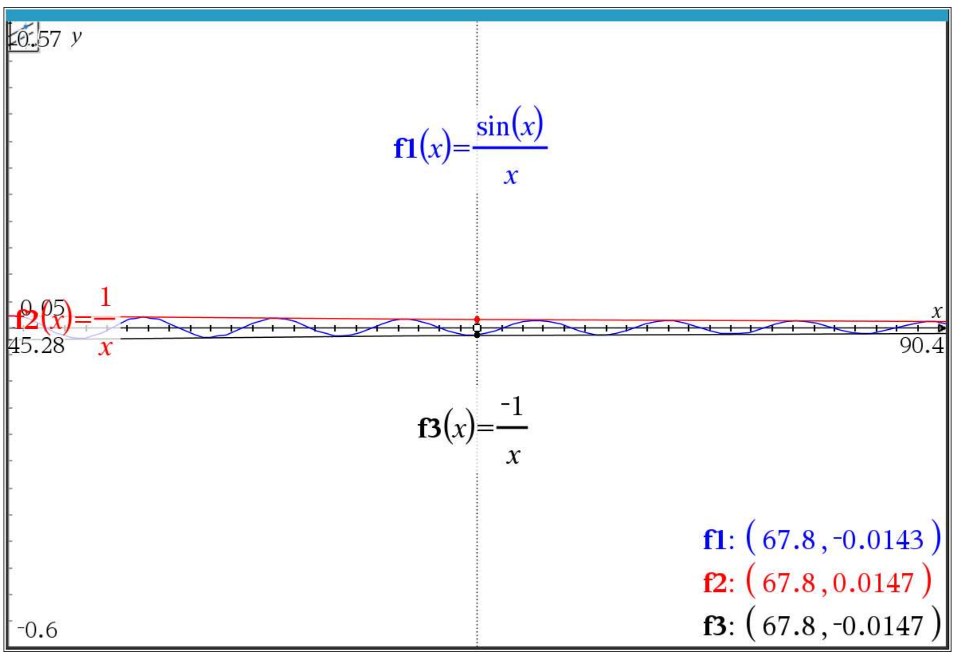
Evaluate $\lim_{x\to\infty} \dfrac{\sin x}{x}$.

Solution: Since $-1 \leq \sin x \leq 1$ for all x, then dividing all parts of those inequalities by $x > 0$ yields

$$-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x} \text{ for all } x > 0 \quad \Rightarrow \quad \lim_{x\to\infty} \frac{\sin x}{x} = 0$$

by the Squeeze Theorem, since $\lim_{x\to\infty} \dfrac{-1}{x} = 0 = \lim_{x\to\infty} \dfrac{1}{x}$.





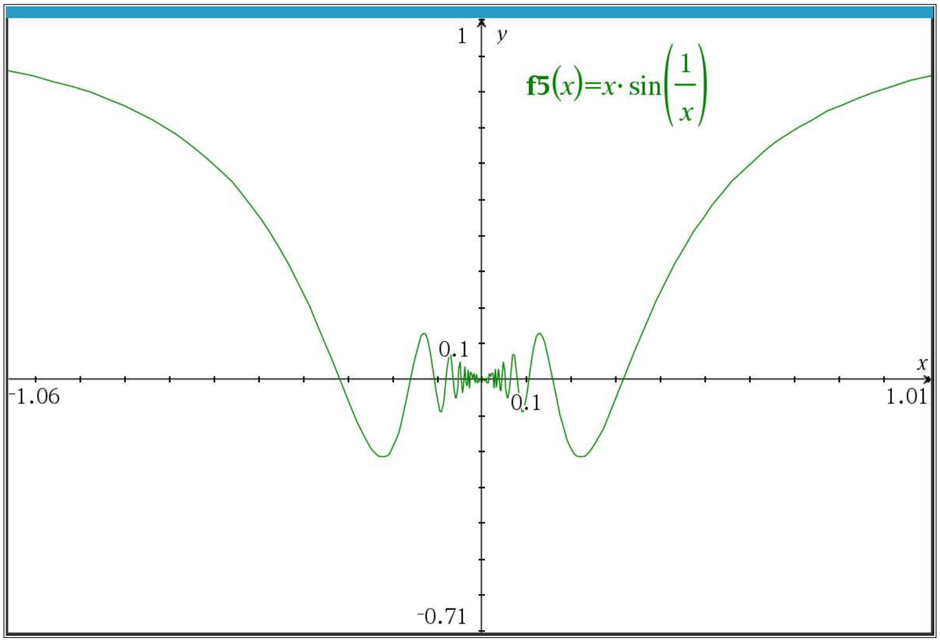
3.2

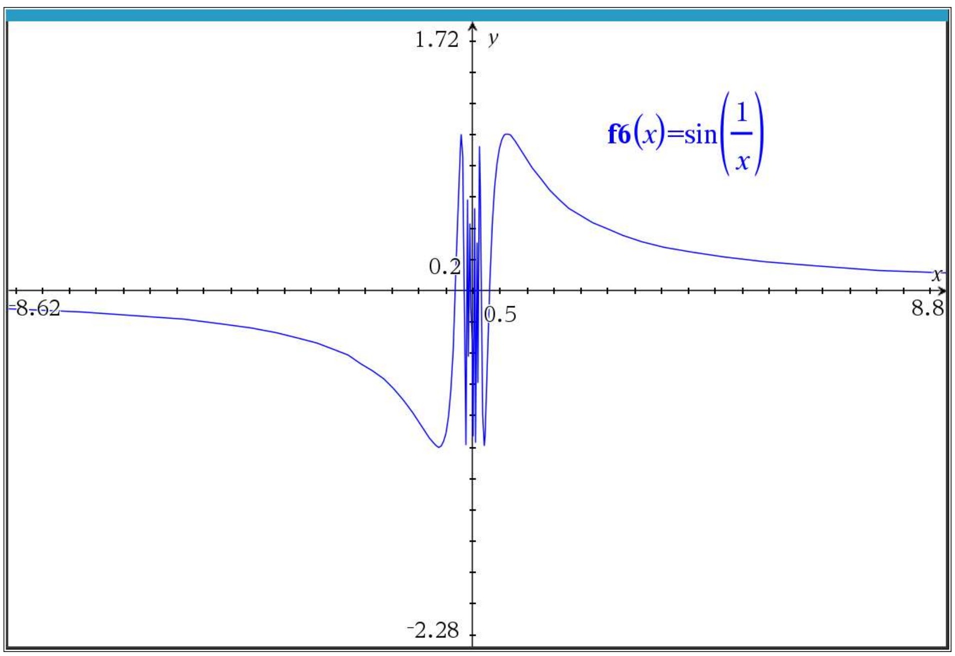
For Exercises 1-18 evaluate the given limit.

12. $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$

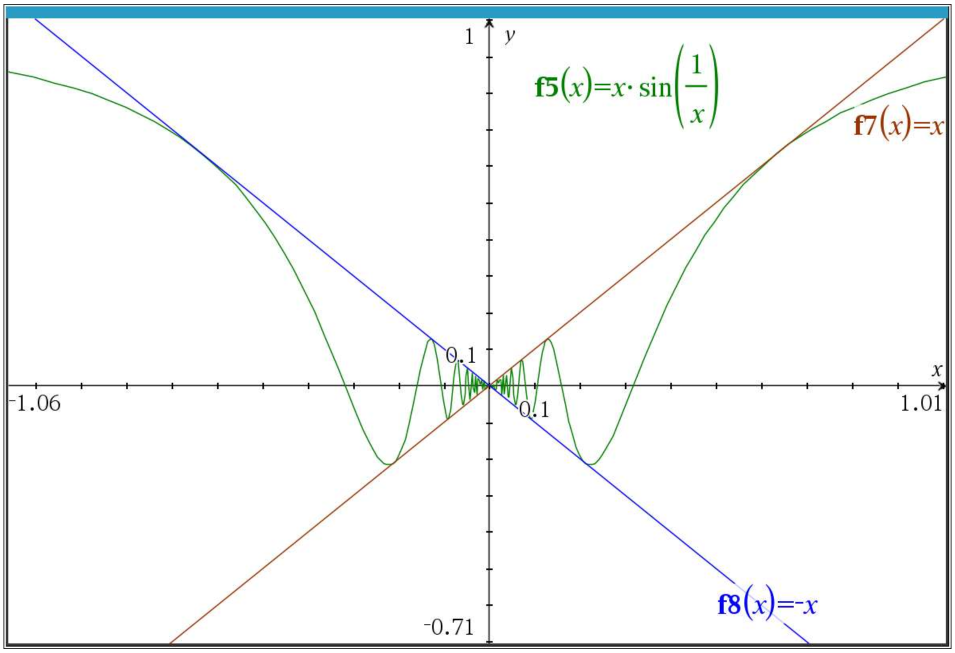
Sin(1/x) does not approach 0 or infinity, so
L'Hopital's rule does not apply.

The graph suggests that the limit is zero,





Guess bounding functions.



can we prove $-x \le x \sin\left(\frac{1}{x}\right) \le x$?

$$-x = (x)(-1) \le x \sin\left(\frac{1}{x}\right) \le (x)(1) = x$$

More examples of epsilon-delta

Find $\lim_{x \to 4}(2x+1)$ and verify the limit with the formal definition of limit.

$= 2(4)+1 = 8+1 = \boxed{9}$

Let $\varepsilon > 0$

Find $\delta > 0$ such that

$0 < |x-4| < \delta \Rightarrow |(2x+1)-9| < \varepsilon$

$|(2x+1)-9| < \varepsilon$

$$|(2x+1)-9| < \varepsilon$$

$$\iff |2x-8| < \varepsilon$$

$$\iff 2|x-4| < \varepsilon$$

$$\iff |x-4| < \frac{\varepsilon}{2}$$

Let $\delta = \frac{\varepsilon}{2}$

$$|x-4| < \delta$$

$$\Rightarrow |x-4| < \frac{\varepsilon}{2}$$

$$\Rightarrow 2|x-4| < \varepsilon$$

$$\Rightarrow |2x-8| < \varepsilon$$

$$\Rightarrow |(2x+1)-9| < \varepsilon$$

Find $\lim_{x \to 1} (x^2+1)$ verify with ε-δ

$$\lim_{x \to 1} (x^2+1) = 1^2+1 = \boxed{2}$$

Let $\varepsilon > 0$ Find $\delta > 0$ such that

$$0 < |x-1| < \delta \Rightarrow |(x^2+1)-2| < \varepsilon$$

$$|(x^2+1)-2| < \varepsilon$$

$|(x+1)^2 - 1|$

$\Leftrightarrow |x^2 - 1| < \varepsilon$

$\Leftrightarrow |x+1||x-1| < \varepsilon$

Let $\boxed{\delta_1 = 1}$

That is assume $|x-1| < 1$

Use this to get a bound on $|x+1|$

$|x-1| < 1$

$\Leftrightarrow -1 < x-1 < 1$

$2-1 < x-1+2 < 1+2$

$1 < x+1 < 3$

$\Rightarrow |x+1| < 3$

$|x+1||x-1| < 3|x-1|$

We will be done if

$3|x-1| < \varepsilon$

$\Leftrightarrow \boxed{|x-1| < \frac{\varepsilon}{3}}$

To cover both conditions

let $\delta = \min(1, \frac{\varepsilon}{3})$

let $\delta = \min\left(1, \frac{\varepsilon}{3}\right)$

You are responsible for using epsilon-delta with linear functions

3.3
Memorize

A function f is **continuous** at $x = a$ if

$$\lim_{x \to a} f(x) = f(a)\,. \tag{3.4}$$

A function is continuous on an interval I if it is continuous at every point in the interval. For a closed interval $I = [a,b]$, a function f is continuous on I if it is continuous on the open interval (a,b) and if $\lim_{x\to a+} f(x) = f(a)$ (i.e. f is **right continuous** at $x = a$) and $\lim_{x\to b-} f(x) = f(b)$ (i.e. f is **left continuous** at $x = b$). A function is **discontinuous** at a point if it is not continuous there. A continuous function is one that is continuous over its entire domain.