02-23-26 MTH 263

2.4 General Exponential and Logarithmic Functions
page 65: 1, 3, 5, 7

3 Topics in Differential Calculus
3.1 Tangent Lines
page 71: 1, 5, 11, 15, 23, 27

3.2 Limits: Formal Definition
page 82: 1, 3, 5, 7, 13

Exam 1		stem & leaf		
56.16667	mean			A-0
20.88424	st.dev			B - 2
59	median	8	48	C - 0
21	min	7		D -3
88	max	6	299	F - 7
12	count	5	399	
		4	8	
		3	6	
		2	16	

2.4: 1
For Exercises 1-9, find the derivative of the given function.

1. $y = \dfrac{3^x + 3^{-x}}{2}$

$$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}\left(3^x + 3^{-x}\right) = \frac{1}{2}\frac{d}{dx}\left(3^x\right) + \frac{1}{2}\frac{d}{dx}\left(3^{-x}\right)$$

$$\frac{d}{dx}\left(a^x\right) = (\ln a)\, a^x$$

$$\frac{d}{dx}\left(a^u\right) = (\ln a)\, a^u \cdot \frac{du}{dx}$$

$$= \frac{1}{2}\left((\ln 3)\,3^x - (\ln 3)\,3^x\right)$$

from chain rule

from chain rule

2.4: 5

5. $y = \log_2(x^2+1)$

$$\frac{d}{dx}(\log_a u) = \frac{1}{u \ln a} \cdot \frac{du}{dx} = \frac{u'}{u \ln a}$$

$a = 2$

$u(x) = x^2+1$

$\frac{d}{dx}\left(\log_2(x^2+1)\right)$

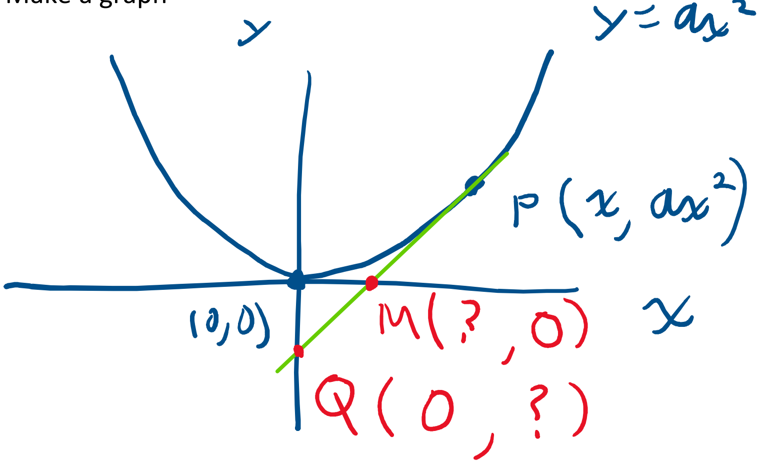
$= \frac{1}{(x^2+1)\ln 2}(2x)$

$= \boxed{\frac{2x}{(x^2+1)\ln 2}}$

3.1: 27

27. For a constant $a > 0$, let P be a point on the curve $y = ax^2$, and let Q be the point where the tangent line to the curve at P intersects the y-axis. Show that the x-axis bisects the line segment $\overline{PQ}$.

Make a graph



Let M = intersection point of tangent line and y-axis

Let M = intersection point
and y-axis

show $\overline{QM} = \overline{PM}$

Find coordinates of P, M, and Q

Finish this at home. We will discuss this next class.

3.2

Memorized informal definition

A real number L is the limit of $f(x)$ as x approaches a if the values of $f(x)$ can be made *arbitrarily* close to L by picking values of x *sufficiently* close to a.

Memorize

Let L and a be real numbers. Then L is the **limit** of a function $f(x)$ as x approaches a, written as

$$\lim_{x\to a} f(x) = L\,,$$

if for any given number $\epsilon > 0$, there exists a number $\delta > 0$, such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x-a| < \delta\,.$$

$\forall$ = for any $\qquad$ $\Rightarrow$ = implies

$\exists$ = there exists $\qquad$ $\Leftrightarrow$ equivalent or bidirectional implies

$\forall \varepsilon > 0,\ \exists\, \delta > 0$ such that

$0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

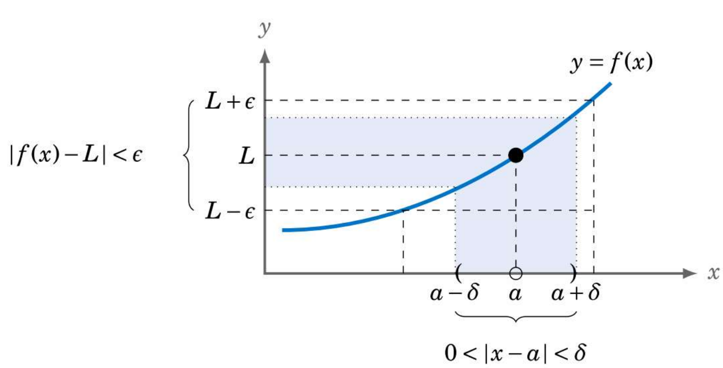


Figure 3.2.1 $\lim_{x\to a} f(x) = L$

Example 3.6

Show that $\lim_{x \to a} x = a$ for any real number a.

Let $\varepsilon > 0$

Find $\delta > 0$ such that

$0 < |x - a| < \delta \Rightarrow |x - a| < \varepsilon$

Let $\delta = \varepsilon$

If $\delta = \varepsilon$, then $0 < |x - a| < \delta \Rightarrow |x - a| < \varepsilon$

Memorize

Call L the **right limit** of a function $f(x)$ as x approaches a, written as

$$\lim_{x \to a+} f(x) = L \,,$$

if $f(x)$ approaches L as x approaches a for values of x larger than a.
Call L the **left limit** of a function $f(x)$ as x approaches a, written as

$$\lim_{x \to a-} f(x) = L \,,$$

if $f(x)$ approaches L as x approaches a for values of x smaller than a.

Memorize

The limit of a function exists if and only if both its right limit and left limit exist and are equal:

$$\lim_{x \to a} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to a-} f(x) = L = \lim_{x \to a+} f(x)$$

$\Leftrightarrow$

iff

if and only if

is equivalent to

Memorize

For a real number a, the limit of a function $f(x)$ **equals infinity** as x approaches a, written as

$$\lim_{x \to a} f(x) = \infty,$$

if $f(x)$ grows without bound as x approaches a, i.e. $f(x)$ can be made larger than any positive number by picking x sufficiently close to a:
For any given number $M > 0$, there exists a number $\delta > 0$, such that

$$f(x) > M \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

$\forall M > 0$, $\exists \delta > 0$ such that

$0 < |x - a| < \delta \Rightarrow f(x) > M$

"close to ∞"

Memorize

For a real number a, the limit of a function $f(x)$ **equals negative infinity** as x approaches a, written as

$$\lim_{x \to a} f(x) = -\infty,$$

if $f(x)$ grows negatively without bound as x approaches a, i.e. $f(x)$ can be made smaller than any negative number by picking x sufficiently close to a:
For any given number $M < 0$, there exists a number $\delta > 0$, such that

$$f(x) < M \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

Memorize

For a real number L, the limit of a function $f(x)$ equals L as x approaches ∞, written as

$$\lim_{x \to \infty} f(x) = L,$$

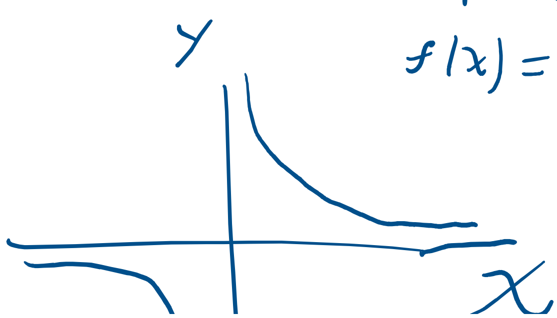
if $f(x)$ can be made arbitrarily close to L for x sufficiently large and positive:
For any given number $\epsilon > 0$, there exists a number $N > 0$, such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad x > N.$$

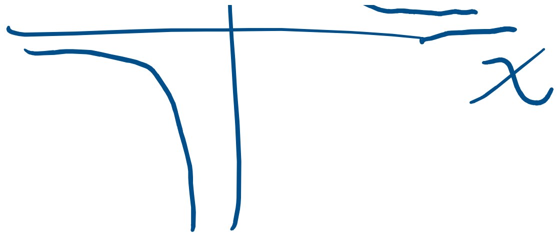
$\forall \varepsilon > 0$, $\exists N > 0$ such that

$x > N \Rightarrow |f(x) - L| < \varepsilon$

$f(x) = \frac{1}{x}$



$\lim_{x \to \infty} \frac{1}{x} = 0$



$\lim_{x\to\infty} \frac{1}{x} = 0$

Let $\varepsilon > 0$, Find $N > 0$ such that

$x > N \Rightarrow \left| \frac{1}{x} - 0 \right| < \varepsilon$

work backwards

$$\left| \frac{1}{x} - 0 \right| < \varepsilon$$

$$\Leftrightarrow \left| \frac{1}{x} \right| < \varepsilon$$

$$\Leftrightarrow \frac{1}{x} < \varepsilon \quad \text{for } x > 0$$

$$\Leftrightarrow \frac{1}{\varepsilon} < x$$

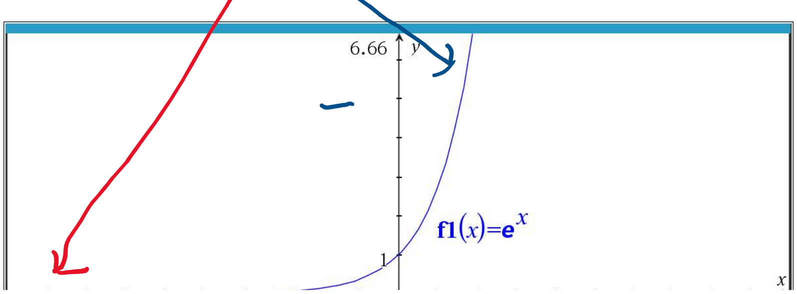
Let $N = \frac{1}{\varepsilon}$

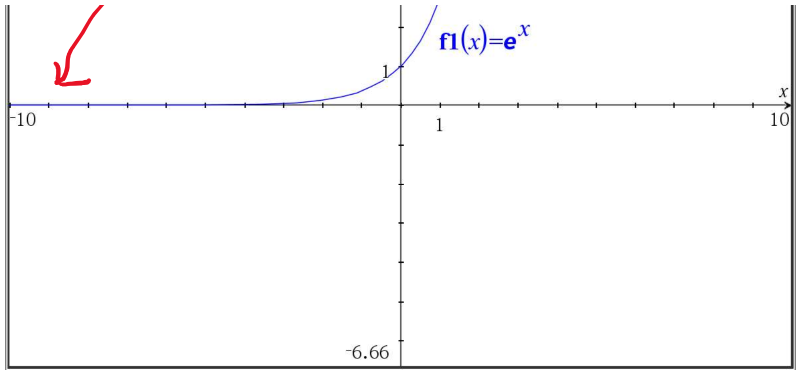
Memorize

$$\lim_{x\to\infty} x^n = \begin{cases} \infty & \text{for any real } n > 0 \\ 0 & \text{for any real } n < 0 \end{cases} \qquad \lim_{x\to-\infty} x^n = \begin{cases} \infty & \text{for } n = 2,4,6,8,\ldots \\ -\infty & \text{for } n = 1,3,5,7,\ldots \end{cases}$$

$$\lim_{x\to\infty} e^x = \infty \qquad \lim_{x\to-\infty} e^x = 0 \qquad \lim_{x\to\infty} e^{-x} = 0 \qquad \lim_{x\to-\infty} e^{-x} = \infty$$

$$\lim_{x\to\infty} \ln x = \infty \qquad \lim_{x\to 0+} \ln x = -\infty$$





Big-O not required for this class.

Memorize

L'Hôpital's Rule: If f and g are differentiable functions and

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty} \text{ or } \frac{0}{0}$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

The number a can be real, ∞, or $-\infty$.

$$\lim_{x\to\infty} \frac{x}{x} = \lim_{x\to\infty} 1 = \boxed{1}$$

$$\left[\frac{\infty}{\infty}\right]$$

$$\lim_{x\to\infty} \frac{x^2}{x} = \lim_{x\to\infty} x = \infty \text{ [dne]}$$

$$\left[\frac{\infty}{\infty}\right]$$

$$\lim_{x\to\infty} \frac{x}{x^2} = \lim_{x\to\infty} \frac{1}{x} = 0$$

$$x \to \infty \qquad x^2 \qquad x \to \infty \quad x$$

$$\left[\frac{\infty}{\infty}\right]$$

The above functions are called "indeterminates"

Use L'Hopital's rule

$$\lim_{x\to\infty} \frac{x}{x} \underset{L}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}(x)}{\frac{d}{dx}(x)} = \lim_{x\to\infty} \frac{1}{1}$$

$$\left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to\infty} 1 = \boxed{1}$$

$$\lim_{x\to\infty} \frac{x^2}{x} \underset{L}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}(x^2)}{\frac{d}{dx}(x)} = \lim_{x\to\infty} \frac{2x}{1}$$

$$\left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to\infty} 2x = \infty \text{ (dne)}$$
