2.3 The Exponential and Natural Logarithm Functions
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Exam 1 - Wednesday - 02/18/26; 1.1-1.6, 2.1-2.3

2.4 General Exponential and Logarithmic Functions
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2.3: 19

19. Suppose it takes 8 hours for 30% of a radioactive substance to decay. Find the half-life of the substance.

$$y(t) = A_0 e^{kt}$$

y(t) = the amount of radioactive substance remaining at time t.
A_0 = the initial amount
k = decay contant

$$y(0) = A_0 e^{(k)(0)} = A_0 e^0 = (A_0)(1) = A_0$$

$$\therefore y(0) = A_0$$

$$y(8) = 70\% A_0 = \frac{7A_0}{10}$$

find t_H (half-life)

$$\boxed{y(t_H) = \frac{A_0}{2}}$$

$$y(8) = A_0 e^{k8} = \frac{7A_0}{10}$$

$$\Rightarrow e^{k8} = \frac{7}{10}$$

$$\ln(e^{k8}) = \ln\left(\frac{7}{10}\right)$$

$$8k = \ln\left(\frac{7}{10}\right)$$

$$\boxed{k = \frac{\ln\left(\frac{7}{10}\right)}{8}}$$

$$y(t) = A_0 e^{\frac{\ln\left(\frac{7}{10}\right)}{8} t}$$

$$y(t) = A_0 e^{\frac{\ln(\frac{7}{10})}{8} t}$$

$$y(t_H) = A_0 e^{\frac{\ln(\frac{7}{10})}{8} t_H} = \frac{A_0}{2}$$

$$\Rightarrow e^{\frac{\ln(\frac{7}{10})}{8} t_H} = \frac{1}{2}$$

$$\Rightarrow \ln\left(e^{\frac{\ln(\frac{7}{10})}{8} t_H}\right) = \ln\left(\frac{1}{2}\right)$$

$\frac{\ln\left(\frac{7}{10}\right)}{8}$ ▸Decimal	$^{-}0.044584$

$$\ln\left(e^{-0.044584 t_H}\right) = \ln\left(\frac{1}{2}\right)$$

$$-0.044584 t_H = \ln\left(\frac{1}{2}\right)$$

$$t_H = \frac{\ln\left(\frac{1}{2}\right)}{-0.044584}$$

$\frac{^{-}8 \cdot \ln(2)}{\ln\left(\frac{7}{10}\right)}$ ▸Decimal	15.5469

The half-life is about 15.55 hours

$$\ln\left(\frac{1}{2}\right) = \ln(1) - \ln(2) = 0 - \ln(2) = -\ln(2)$$

$$y(t) = A_0 e^{kt}$$

$$\frac{dy}{dt} = A_0 e^{kt} k$$

$$\boxed{\frac{dy}{dt} = k A_0 e^{kt}}$$

$$\boxed{\frac{dy}{dt} = k y(t)}$$

Thus, y(t) is a solution to the above differential equation.

2.3: 13

13. Show that $\frac{d}{dx}(\ln(kx)) = \frac{1}{x}$ for all constants $k > 0$.

$$\frac{d}{dx}(\ln(kx)) = \left(\frac{1}{kx}\right)\frac{d}{dx}(kx)$$

$$= \left(\frac{1}{kx}\right)(k)$$

$$= \frac{1}{x} \quad \checkmark$$

$$\frac{d}{dx}(\ln(kx)) = \frac{d}{dx}(\ln k + \ln x)$$

$$= 0 + \frac{1}{x}$$

$$= \frac{1}{x} \quad \checkmark$$

solve

$$\log_2 x + \log_2 (x+1) = 2$$

$$\log_2 (x(x+1)) = 2$$

$$2^2 = x(x+1)$$

$$x^2 + x = 4$$

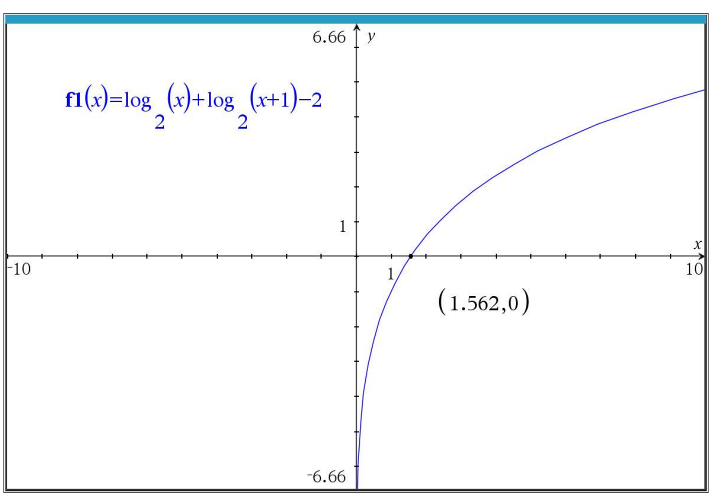
$$x^2 + x - 4 = 0$$

$$x = \frac{-1 \pm \sqrt{1-(4)(1)(-4)}}{2(1)}$$

$$x = \frac{-1 \pm \sqrt{17}}{2}$$

$x > 0$

$$\Rightarrow \boxed{x = \frac{-1+\sqrt{17}}{2}}$$



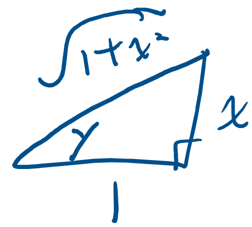
(-1+sqrt(17))/2=1.56155281280883

Prove $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$

Let $y = \arctan(x)$

$$\tan y = x = \frac{x}{1}$$

Find $\frac{dy}{dx}$



$$\frac{d}{dx}(\tan y) = \frac{dx}{dx}$$

$$(\sec^2 y)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y$$

$$= \left(\frac{1}{\sqrt{1+x^2}}\right)^2$$

$$= \frac{1}{1+x^2} \quad \checkmark$$

$$x = \tan y$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\sec^2 y} = \frac{1}{1+x^2} \text{ (see above } \triangle\text{)}$$

$$\frac{dx}{dy} = \frac{d}{dy}(\tan y) = \sec^2 y$$

Brief overview of 2.4, 3.1

After class notes

1.3

For Exercises 1-9, let dx be an infinitesimal and prove the given formula.

1.3

For Exercises 1-9, let dx be an infinitesimal and prove the given formula.

$(dx)^2 = 0$

1. $(dx + 1)^2 = 2dx + 1$

$(dx+1)(dx+1)$

$= (dx)^2 + 2dx + 1$

$= 0 + 2dx + 1$

$= 2dx + 1$ ✓

Prove

5. $\sin 2dx = 2dx$

Given

$\boxed{\sin dx = dx}$

$\boxed{\cos dx = 1}$

Remember $\sin(2\theta) = 2\sin\theta\cos\theta$

$= 2\sin dx \cos dx$

$= 2(dx)(1)$

$= 2dx$ ✓

1.4

For Exercises 1-14, use the rules from this section to find the derivative of the given function.

5. $f(x) = x \sin x$

$f'(x) = \left(\frac{d}{dx}(x)\right)\sin x + x\frac{d}{dx}(\sin x)$

$= (1)\sin x + x\cos x$

$= \boxed{\sin x + x\cos x}$

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6. $f(x) = x^2 \tan x$

$f'(x) = (\tan x)\frac{d}{dx}(x^2) + x^2\frac{d}{dx}(\tan x)$

$= (\tan x)(2x) + x^2\sec^2 x$

$= \boxed{2x\tan x + x^2\sec^2 x}$

$$= \boxed{2x \tan x + x^2 \sec^2 x}$$

1.3

For Exercises 1-9, let dx be an infinitesimal and prove the given formula.

$$\boxed{(dx)^2 = 0}$$

3. $(dx + 1)^{-1} = 1 - dx$

$$= \frac{1}{dx+1}$$

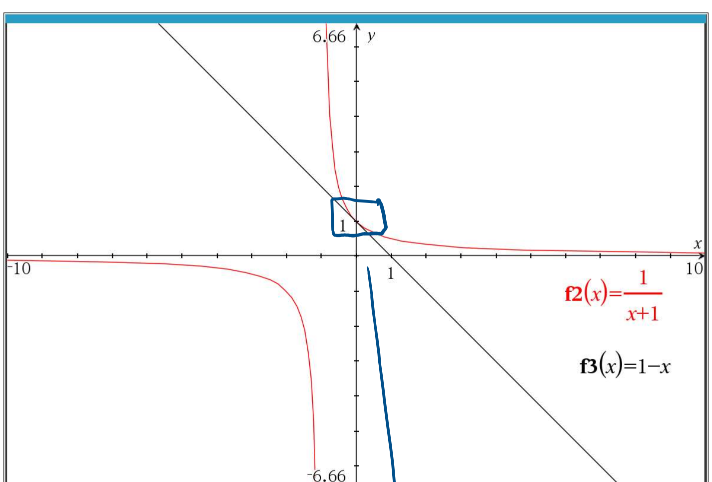
$$= \left(\frac{1}{1+dx}\right)\left(\frac{1-dx}{1-dx}\right)$$

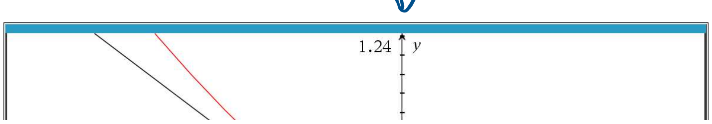
$$= \frac{1-dx}{1^2 - (dx)^2}$$

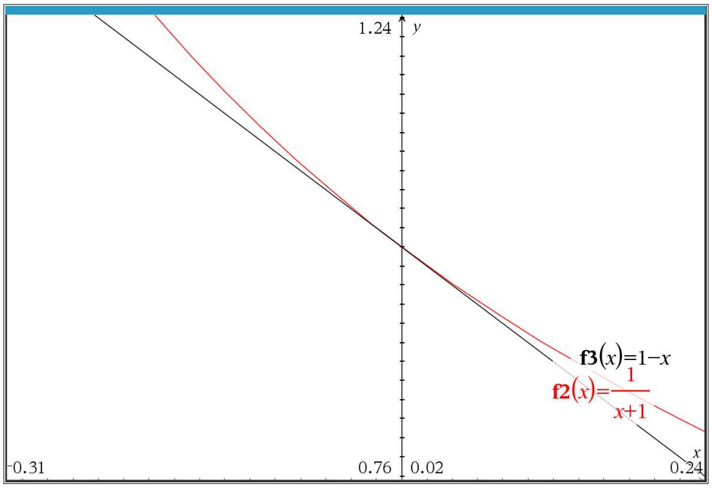
$$= \frac{1-dx}{1-0} = \frac{1-dx}{1} = 1-dx \quad \checkmark$$

$(a+b)(a-b)$
$= a^2 - b^2$

for small x, $\frac{1}{x+1} \approx 1-x$







1.4

For Exercises 1-14, use the rules from this section to find the derivative of the given function.

3. $f(x) = \dfrac{2x^6}{3} - \dfrac{3}{2x^6}$

$$f'(x) = \frac{d}{dx}\left(\frac{2x^6}{3}\right) - \frac{d}{dx}\left(\frac{3}{2x^6}\right)$$

$$= \frac{d}{dx}\left(\left(\frac{2}{3}\right)x^6\right) - \frac{d}{dx}\left(\left(\frac{3}{2}\right)\frac{1}{x^6}\right)$$

$$= \left(\frac{2}{3}\right)\frac{d}{dx}\left(x^6\right) - \frac{3}{2}\frac{d}{dx}\left(x^{-6}\right)$$

$$= \left(\frac{2}{3}\right)\left(6x^5\right) - \left(\frac{3}{2}\right)(-6)\left(x^{-7}\right)$$

$$= \boxed{4x^5 + 9x^{-7}}$$

$$\frac{d}{dx}\left(kf(x)\right) = k\frac{df}{dx}$$

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

2.3

For Exercises 1-12, find the derivative of the given function.

2. $y = xe^{x^2}$

$$\frac{dy}{dx} = x\frac{d}{dx}\left(e^{x^2}\right) + e^{x^2}\frac{d}{dx}(x)$$

$$= x\left(2xe^{x^2}\right) + e^{x^2}(1)$$

$$= \boxed{e^{x^2}\left(2x^2+1\right)}$$