02-09-26 MTH 263

1.6 Higher Order Derivatives
page 45: 1, 3, 7, 9

2 Derivatives of Common Functions

2.1 Inverse Functions
page 50: 1, 3, 5

2.2 Trigonometric Functions and Their Inverses
page 54: 1, 4, 5, 15

1.6: 7

7. Find the first five derivatives of $f(x) = \sin x$. Use those to find $f^{(100)}(x)$ and $f^{(2014)}(x)$.

$f^{0}(x) = f(x) = \sin x$
$f'(x) = \cos x$
$f''(x) = -\sin x$
$f'''(x) = -\cos x$
$f^{(4)}(x) = \sin x$
$f^{(5)}(x) = \cos x$

$f^{(6)}(x) = -\sin x$
$f^{(7)}(x) = -\cos x$
$f^{(8)}(x) = \sin x$
$f^{(9)}(x) = \cos x$
$f^{(10)}(x) = -\sin x$

$f^{(0)}(x) = f^{(4)}(x) = f^{(8)}(x) = \sin x$

$\frac{100}{4} = 25$

$\Rightarrow f^{(100)}(x) = \sin x$

$2014 \div 4 = 503$: 20, 14, 12, remainder 2

$\Rightarrow f^{(2014)}(x) = -\sin(x)$

$\frac{d^{2014}}{dx^{2014}}(\sin(x))$ $-\sin(x)$

$\frac{d^{100}}{dx^{100}}(\sin(x))$ $\sin(x)$

$\frac{d^{2014}}{dx^{2014}}(\sin(x))$ $-\sin(x)$

2.1
Supplied

To obtain a formula in prime notation for the derivative of an inverse function, notice that for all x in the domain of an invertible differentiable function f,

Remember

$$f^{-1}(f(x)) = x \quad \Rightarrow \quad \frac{d}{dx}\left(f^{-1}(f(x))\right) = \frac{d}{dx}(x) \quad \Rightarrow \quad \left(f^{-1}\right)'(f(x)) \cdot f'(x) = 1$$

by the Chain Rule, and hence:

$$\left(f^{-1}\right)'(f(x)) = \frac{1}{f'(x)} \qquad \text{if } f'(x) \neq 0$$

Two equivalent ways to write this are:

$$\left(f^{-1}\right)'(c) = \frac{1}{f'(a)} \qquad \text{where } c = f(a) \text{ and } f'(a) \neq 0$$

and

$$\left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))} \qquad \text{if } f'(f^{-1}(x)) \neq 0$$

memorize

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{(\cos x)(\cos x) - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \boxed{\sec^2 x}$$

supplied

function	$\sin^{-1} x$	$\cos^{-1} x$	$\tan^{-1} x$	$\csc^{-1} x$	$\sec^{-1} x$	$\cot^{-1} x$
domain	$[-1,1]$	$[-1,1]$	$(-\infty,\infty)$	$\lvert x\rvert \ge 1$	$\lvert x\rvert \ge 1$	$(-\infty,\infty)$
range	$[-\frac{\pi}{2},\frac{\pi}{2}]$	$[0,\pi]$	$(-\frac{\pi}{2},\frac{\pi}{2})$	$(-\frac{\pi}{2},0)\cup(0,\frac{\pi}{2})$	$(0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi)$	$(0,\pi)$

Memorize

[1]The *arc* notation arcsin x, arccos x, arctan x, arccsc x, arcsec x, arccot x is often used in place of $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\csc^{-1} x$, $\sec^{-1} x$, $\cot^{-1} x$, respectively.

Supplied or be able to derive

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \quad (\text{for } |x|<1) \qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}} \quad (\text{for } |x|>1)$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \quad (\text{for } |x|<1) \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}} \quad (\text{for } |x|>1)$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

For the derivative of $\cos^{-1} x$, recall that $y = \cos^{-1} x$ is an *angle* between 0 and π radians, defined for $-1 \le x \le 1$. Since $\cos y = x$ by the definition of y, then $\frac{dx}{dy} = -\sin y$ and

$$\sin^2 y = 1 - \cos^2 y = 1 - x^2 \quad \Rightarrow \quad \sin y = \pm\sqrt{1-x^2} = \sqrt{1-x^2}$$

since $0 \le y \le \pi$ (which means $\sin y$ must be nonnegative). Thus:

$$\frac{d}{dx}(\cos^{-1} x) = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{-\sin y} = -\frac{1}{\sqrt{1-x^2}} \quad \checkmark$$

$$y = \cos^{-1} x \Rightarrow \boxed{\cos y = x}$$

Implicit differentiation

Find $\frac{d}{dx}(\cos^{-1} x)$

Let $y = \cos^{-1}(x)$ [Goal: find $\frac{dy}{dx}$]

$\Rightarrow x = \cos y(x)$

Note $y = y(x)$ is a function of x

$$\frac{d}{dx}(x) = \frac{d}{dx}(\cos y(x))$$

$$1 = (-\sin y)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y}$$

$$x = \cos y$$

$$\Rightarrow x^2 = \cos^2 y = 1 - \sin^2 y$$

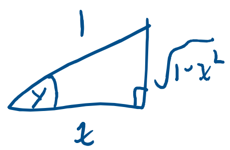
$$\Rightarrow \sin^2 y = 1 - x^2$$

$$\Rightarrow \sin y = \pm\sqrt{1-x^2}$$

$$y \geq 0$$

$$\Rightarrow \sin y = \sqrt{1-x^2}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$



$$x = \cos y$$

$$\frac{x}{1} = \frac{adj}{hyp}$$

$$\sin y = \frac{opp}{hyp} = \sqrt{1-x^2}$$

Let $f(x) = x^3$

Find $\frac{d}{dx}\left(f^{-1}(x)\right)$

Let $y = f^{-1}(x) = x^{\frac{1}{3}}$

$$\frac{dy}{dx} = \left(\frac{1}{3}\right) x^{-2/3}$$

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{3x^{2/3}}$$

2.1

For Exercises 1-8, show that the given function $y = f(x)$ is one-to-one over the given interval, then find the formulas for the inverse function f^{-1} and its derivative. Use Example 2.2 as a guide, including putting f^{-1} and its derivative in terms of x.

7. $f(x) = \frac{1}{x^2}$, for all $x > 0$

Show f is 1-1

Assume $\frac{1}{c^2} = \frac{1}{d^2}$

$$\Rightarrow c^2 = d^2$$

$$\Rightarrow c = \pm d$$

$$x > 0$$

$x > 0$

$\Rightarrow c = d$

$\therefore f$ is 1-1 on $(0, \infty)$

$\therefore f^{-1}$ exists

Let $y = f^{-1}(x)$

Find $\frac{dy}{dx}$

$f(y) = x$

Find $f^{-1}(x)$

y has separate meaning here

$y = f(x) = \frac{1}{x^2}$

$x = \frac{1}{y^2}$

$y^2 = \frac{1}{x}$

$y = \frac{1}{\sqrt{x}}$ because $\frac{1}{x^2} > 0$ all $x > 0$

$$\boxed{f^{-1}(x) = \frac{1}{\sqrt{x}}}$$

$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{d}{dx}\left(x^{-\frac{1}{2}}\right)$

$= \left(-\frac{1}{2}\right)\left(x^{-\frac{3}{2}}\right)$

1

$$= -\frac{1}{2x^{3/2}}$$

7. $f^{-1}(x) = \frac{1}{\sqrt{x}}$, $\left(f^{-1}\right)'(x) = -\frac{1}{2}x^{-3/2}$

2.1

For Exercises 1-16, find the derivative of the given function $y = f(x)$.

8. $y = \cos^{-1}(\sin x)$

Find $\frac{dy}{dx}$

$\cos y = \sin x$

$\frac{d}{dx}(\cos y) = \frac{d}{dx}(\sin x)$

$(-\sin y)\frac{dy}{dx} = \cos x$

$\frac{dy}{dx} = \frac{\cos x}{-\sin y}$

$\cos y = \sin x$

$\cos y = \sqrt{1-\sin^2 y}$

$\sin x = \sqrt{1-\sin^2 y}$

$\sin^2 x = 1 - \sin^2 y$

$\sin^2 y = 1 - \sin^2 x$

$\sin y = \sqrt{1-\sin^2 x}$

$$\sin y = \sqrt{1-\sin^2 x}$$

$$\frac{dy}{dx} = \frac{\cos x}{-\sqrt{1-\sin^2 x}}$$

$$\frac{dy}{dx} = -\frac{\cos x}{\cos x}$$

$$\frac{dy}{dx} = -1$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \quad (\text{for } |x| < 1)$$

$$\frac{d}{dx} \cos^{-1}(\sin x)$$

$$= -\frac{1}{\sqrt{1-\sin^2 x}} \cdot \cos x$$

$$= -\frac{1}{\cos x} \cos x$$

$$= -1$$