

1.5 The Chain Rule

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1.6 Higher Order Derivatives

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2 Derivatives of Common Functions

2.1 Inverse Functions

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1.6

Memorize

Notation for the second derivative of $y = f(x)$: The following are all equivalent:

$$f''(x) , f^{(2)}(x) , \frac{d^2y}{dx^2} , \frac{d^2}{dx^2}(f(x)) , y'' , y^{(2)} , \ddot{y} , \ddot{f}(x) , \frac{d^2f}{dx^2} , D^2f(x)$$

Notation for the n -th derivative of $y = f(x)$: The following are all equivalent:

$$f^{(n)}(x) , \frac{d^ny}{dx^n} , \frac{d^n}{dx^n}(f(x)) , y^{(n)} , \frac{d^nf}{dx^n} , D^n f(x)$$

memorize

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx} \left(\frac{d^2y}{dx^2} \right) = \frac{d^2}{dx^2} \left(\frac{dy}{dx} \right)$$

⋮

$$\frac{d^ny}{dx^n} = \frac{d}{dx} \left(\frac{d^{n-1}y}{dx^{n-1}} \right) = \frac{d^{n-1}}{dx^{n-1}} \left(\frac{dy}{dx} \right)$$

Memorize

$s(t)$ = position at time t

$$v(t) = \text{velocity at time } t \\ = \frac{ds}{dt} = s'(t) = \dot{s}(t)$$

$a(t)$ = acceleration at time t

$$= \frac{dv}{dt} = v'(t) = \dot{v}(t)$$

$$d^2s/dt^2 = \ddot{s}$$

$s(t)$ = position at time t

$v(t)$ = velocity at time t

$$= \frac{ds}{dt} = s'(t) = \dot{s}(t)$$

$a(t)$ = acceleration at time t

$$= \frac{dv}{dt} = v'(t) = \dot{v}(t)$$

$$= \frac{d}{dt} \left(\frac{ds}{dt} \right) = \frac{d^2s}{dt^2} = s''(t) = \ddot{s}(t)$$

memorize

The **zero-th derivative** $f^{(0)}(x)$ of a function $f(x)$ is defined to be the function $f(x)$ itself: $f^{(0)}(x) = f(x)$. There is a way to define **fractional derivatives**, e.g. the *one-half derivative* $f^{(1/2)}(x)$, which will be discussed in Chapter 6.

An immediate consequence of the definition of higher order derivatives is:

$$\frac{d^{m+n}}{dx^{m+n}}(f(x)) = \frac{d^m}{dx^m} \left(\frac{d^n}{dx^n}(f(x)) \right) \quad \text{for all integers } m \geq 0 \text{ and } n \geq 0.$$

Supplied

$$\frac{d^n}{dx^n}(x^n) = n! \quad \text{for all integers } n \geq 0$$

Prove by mathematical induction

Basis step: let $n = 0$.

Assume $x \neq 0$

$$\begin{aligned} \frac{d^0}{dx^0}(x^0) &\stackrel{?}{=} 0! \\ x^0 &\stackrel{?}{=} 1 \\ 1 &= 1 \quad \checkmark \end{aligned}$$

$$\begin{aligned} \text{Let } n=1 \\ \frac{d}{dx}(x^1) &\stackrel{?}{=} 1! \\ \frac{d}{dx}(x) &\stackrel{?}{=} 1 \\ 1 &= 1 \quad \checkmark \end{aligned}$$

$$\begin{aligned} \text{Def} \\ 0! &= 1 \\ n! &= n(n-1)! \\ &= n(n-1)(n-2)\dots 3 \cdot 2 \cdot 1 \end{aligned}$$

$$1 = 1 \quad \checkmark$$

$$\text{Let } n=2 \quad ? = 2!$$
$$\frac{d^2}{dx^2}(x^2) \stackrel{?}{=} 2!$$

$$\frac{d}{dx}\left(\frac{d}{dx}(x^2)\right) \stackrel{?}{=} 2$$

$$\frac{d}{dx}(2x) \stackrel{?}{=} 2$$

$$2 = 2 \quad \checkmark$$

Assume $\frac{d^n}{dx^n}(x^n) = n!$ for any fixed $n \geq 0$

Prove $\frac{d^{n+1}}{dx^{n+1}}(x^{n+1}) = (n+1)!$

$$\frac{d^n y}{dx^n} = \frac{d}{dx} \left(\frac{d^{n-1} y}{dx^{n-1}} \right) = \frac{d^{n-1}}{dx^{n-1}} \left(\frac{dy}{dx} \right)$$

$$\frac{d^{n+1}}{dx^{n+1}}(x^{n+1}) = \frac{d^n}{dx^n} \left(\frac{d}{dx}(x^{n+1}) \right)$$

$$= \frac{d^n}{dx^n}((n+1)x^n)$$

$$= (n+1) \frac{d^n}{dx^n}(x^n)$$

$$= (n+1)n! \quad \text{by Ind hyp}$$

$$= (n+1)! \quad \checkmark$$

$$\frac{d^{n+1}}{dx^{n+1}}(x^{n+1}) = (n+1)! \quad \checkmark$$

$$\begin{aligned}\frac{d^{n+1}}{dx^{n+1}}(x^n) &= \frac{d}{dx} \left(\frac{d^n}{dx^n} x^n \right) \\ &= \frac{d}{dx} (n!) \\ &= 0\end{aligned}$$

supplied

The $(n+1)$ -st derivative ("n plus first derivative") of a polynomial of degree n is 0:
For any polynomial $p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ of degree n , $\frac{d^{n+1}}{dx^{n+1}}(p(x)) = 0$.

Find $\frac{d^3}{dx^3}(5x^2 + 10x - 4)$

$$\frac{d}{dx}(5x^2 + 10x - 4) = 10x + 10$$

$$\frac{d}{dx}(10x + 10) = 10$$

$$\frac{d}{dx}(10) = 0$$

$$\therefore \frac{d^3}{dx^3}(5x^2 + 10x - 4) = 0$$

$$\therefore \frac{d^3}{dx^3} (5x^2 + 10x - 4) = 0$$

$$\begin{aligned} & \frac{d^3}{dx^3} (5x^2 + 10x - 4) \\ &= \frac{d^2}{dx^2} \left(\frac{d}{dx} (5x^2 + 10x - 4) \right) \\ &= \frac{d^2}{dx^2} (10x + 10) \\ &= \frac{d}{dx} \left(\frac{d}{dx} (10x + 10) \right) \\ &= \frac{d}{dx} (10) \\ &= \boxed{0} \quad \checkmark \end{aligned}$$

Gemini

Please give me an example of a function for which we need the 3rd derivative. Don't give me the solution.

In physics and engineering, the most common and intuitive reason to go beyond the second derivative is to study the "**Jerk**" of an object.

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In physics and engineering, the most common and intuitive reason to go beyond the second derivative is to study the "Jerk" of an object. 

While the first derivative gives us velocity and the second gives us acceleration, the third derivative measures how quickly that acceleration is changing. This is crucial for designing roller coasters, elevators, or high-speed trains to ensure a smooth ride—if the jerk is too high, it can cause discomfort or even whiplash.

The Problem

Imagine you are a structural engineer designing a new high-speed elevator for a skyscraper. To ensure passengers don't feel "jolted" when the elevator starts or stops, you need to analyze the change in acceleration over time.

Consider the following position function $s(t)$, which represents the displacement of the elevator (in meters) over time t (in seconds):

$$s(t) = \frac{1}{60}t^5 - \frac{1}{12}t^4 + 10t$$

Your Task

To find the Jerk $j(t)$, you will need to differentiate this function three times with respect to t :

1. **First Derivative:** $v(t) = s'(t)$ (Velocity)
2. **Second Derivative:** $a(t) = s''(t)$ (Acceleration)
3. **Third Derivative:** $j(t) = s'''(t)$ (Jerk)

$$s'(t) = \left(\frac{5}{60}\right)t^4 - \frac{4t^3}{12} + 10$$

$$s'(t) = \frac{t^4}{12} - \frac{t^3}{3} + 10$$

$$s''(t) = \frac{4t^3}{12} - \frac{3t^2}{3} + 0$$

$$s''(t) = \frac{t^3}{3} - t^2$$

$$s''(t) = \frac{t^3}{3} - t^2$$

$$s'''(t) = 3t^2 - 2t$$

$$s'''(t) = t^2 - 2t$$

2.1

Recall that a function f is **one-to-one** (often written as 1-1) if it assigns distinct values of y to distinct values of x . In other words, if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$. Equivalently, f is one-to-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. There is a simple *horizontal rule* for determining whether a function $y = f(x)$ is one-to-one: f is one-to-one if and only if every horizontal line intersects the graph of $y = f(x)$ in the xy -coordinate plane at most once (see Figure 2.1.3).

This is equivalent to $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$$\text{Let } f(x) = 3x + 5$$

is f 1-1?

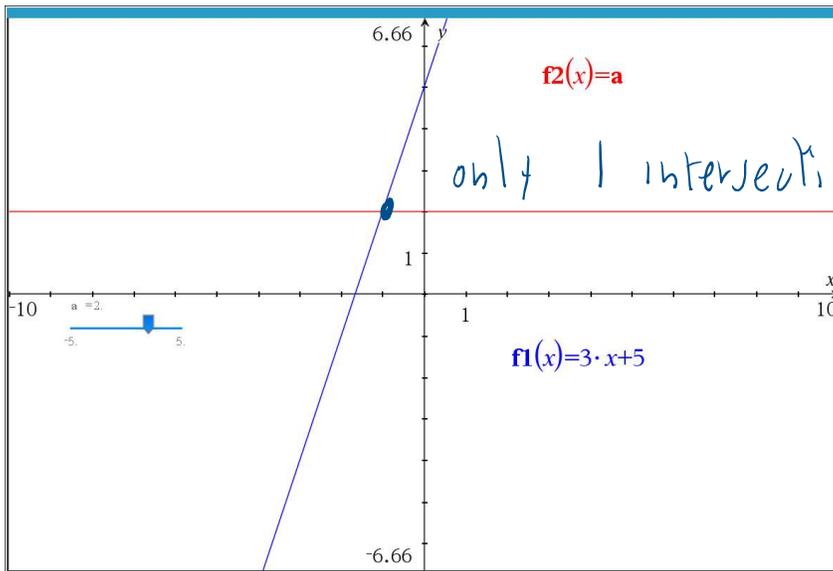
$$\text{Assume } f(x_1) = f(x_2)$$

$$\Rightarrow 3x_1 + 5 = 3x_2 + 5$$

$$\Rightarrow 3x_1 = 3x_2$$

$$\Rightarrow x_1 = x_2$$

$\therefore f$ is 1-1



only 1 intersection point

$\therefore f$ 1-1

Find $f^{-1}(x)$

$$f(x) = 3x + 5$$

$$y = 3x + 5 \quad \text{interchange } x, y$$

$$x = 3y + 5 \quad \text{solve for } y$$

$$3y = x - 5$$

$$y = \frac{x - 5}{3}$$

change y to $f^{-1}(x)$

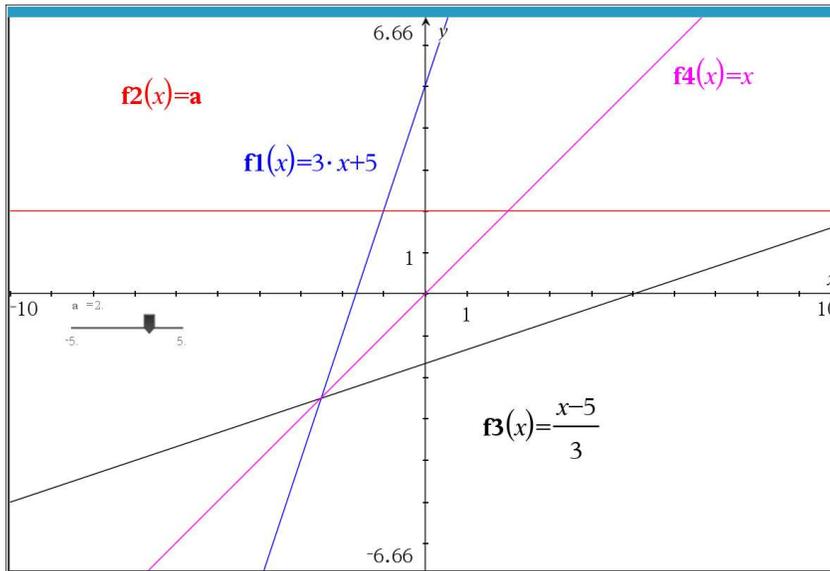
$$f^{-1}(x) = \frac{x - 5}{3}$$

$$(f \circ f^{-1})(x) = f(f^{-1}(x))$$

$$= f\left(\frac{x - 5}{3}\right)$$

$$= 3\left(\frac{x - 5}{3}\right) + 5$$

$$\begin{aligned} & \sim 1 \quad 3 \quad 1 \\ & \sim x - 5 + 5 \\ & = x \end{aligned}$$



The graph of a function and its inverse must be symmetric with respect to the line $y = x$.

Supplied

Derivative of an Inverse Function: If $y = f(x)$ is differentiable and has an inverse function $x = f^{-1}(y)$, then f^{-1} is differentiable and its derivative is

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \quad \text{if } \frac{dy}{dx} \neq 0.$$

Example 2.1

Find the inverse f^{-1} of the function $f(x) = x^3$ then find the derivative of f^{-1} .

Solution: The function $y = f(x) = x^3$ is one-to-one over the set of all real numbers (why?) so it has an inverse function $x = f^{-1}(y)$ defined for all real numbers, namely $x = f^{-1}(y) = \sqrt[3]{y}$.

The derivative of f^{-1} is

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{3x^2}, \text{ which is in terms of } x, \text{ so putting it in terms of } y \text{ yields}$$

$$= \frac{1}{3(\sqrt[3]{y})^2} = \frac{1}{3y^{2/3}}$$

which agrees with the derivative obtained by differentiating $x = \sqrt[3]{y}$ directly. Note that this derivative is defined for all y except $y = 0$, which occurs when $x = \sqrt[3]{0} = 0$, i.e. at the point $(x, y) = (0, 0)$.

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check $f(x)$ is 1-1
Assume $f(c) = f(d)$

$$\Rightarrow c^3 = d^3$$

$$\Rightarrow c = d$$

$\therefore f$ is 1-1
 $\therefore f^{-1}$ exists

$$y = x^3$$

$$x = y^{\frac{1}{3}}$$

$$y = \sqrt[3]{x}$$

$$f^{-1}(x) = \sqrt[3]{x}$$

$$f^{-1}(x) = x^{\frac{1}{3}}$$

$$\frac{d}{dx}(f^{-1}(x)) = \left(\frac{1}{3}\right)x^{\frac{1}{3}-1} \Rightarrow \left(\frac{1}{3}\right)x^{-\frac{2}{3}}$$

$$= \frac{1}{3\sqrt[3]{x^2}}$$