02-02-26 MTH 263

1.3 The Derivatives: Infinitesimal Approach
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1.4 Derivatives of Sums, Products and Quotients
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1.5 The Chain Rule
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Your Name MTH 263 quiz 1

Open homework notebook, closed everything else.
Calculator OK.

1.3: 1

A

For Exercises 1-9, let dx be an infinitesimal and prove the given formula.

1. $(dx + 1)^2 = 2dx + 1$

$(dx)^2 + 2dx + 1$

$= 0 + 2dx + 1$

$= 2dx + 1$ ✓

1.4: 1

A

For Exercises 1-14, use the rules from this section to find the derivative of the given function.

1. $f(x) = x^2 - x - 1$

$f'(x) = 2x - 1$

$\frac{df}{dx} = \frac{d}{dx}(x^2 - x - 1)$

$= \frac{d}{dx}(x^2) + \frac{d}{dx}(-x) + \frac{d}{dx}(-1)$ sum rule

$= 2x - \frac{d}{dx}(x) + 0$

power rule constant multiple rule

$= \boxed{2x - 1}$

1.4 memorize

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Memorize

For $n \geq 1$ differentiable functions $f_1, \ldots, f_n$ and constants $c_1, \ldots, c_n$:

$$\frac{d}{dx}(c_1 f_1 + \cdots + c_n f_n) = c_1 \frac{df_1}{dx} + \cdots + c_n \frac{df_n}{dx} \qquad (1.10)$$

Memorize

Power Rule: $\frac{d}{dx}(x^n) = n\,x^{n-1}$ for any integer n

Let $n=0$

$\frac{d}{dx}(x^0) = \frac{d}{dx}(1) = \boxed{0}$

$x \neq 0$

$0 \cdot x^{0-1} = \boxed{0}$ ✓

True for $n=0$

let $n=1$

$\frac{d}{dx}(x^1) = \frac{d}{dx}(x) = \boxed{1}$

$1 \cdot x^{1-1} = 1 \cdot x^0 = \boxed{1}$ ✓

let $n=2$

$\frac{d}{dx}(x^2) = \frac{d}{dx}(x \cdot x)$

$= x\frac{dx}{dx} + x\frac{dx}{dx}$

$= x(1) + x(1)$

$= \boxed{2x}$

$2x^{2-1} = 2x^1 = \boxed{2x}$

Supplied

Principle of Mathematical Induction

A statement $P(n)$ about integers $n \geq k$ is true for all $n \geq k$ if:

1. $P(k)$ is true.
2. If $P(n)$ is true for some integer $n \geq k$ then $P(n+1)$ is true.

Memorize

In general, the derivative of a polynomial of degree $n \geq 0$ is given by:

For any constants $a_0, \ldots, a_n$ with $n \geq 0$:

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For any constants $a_0, \ldots, a_n$ with $n \geq 0$:

$$\frac{d}{dx}\left(a_n x^n + a_{n-1}x^{n-1} \cdots + a_2 x^2 + a_1 x + a_0\right) = n a_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2 x + a_1$$

Differentiate term-by-term, using constant multiple rule and power rule

1.5

Memorize

Chain Rule: If f is a differentiable function of u, and u is a differentiable function of x, then f is a differentiable function of x, and its derivative with respect to x is:

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

Chain Rule: If g is a differentiable function of x, and f is a differentiable function on the range of g, then $f \circ g$ is a differentiable function of x, and its derivative with respect to x is:

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

A

For Exercises 1-9, let dx be an infinitesimal and prove the given formula.

5. $\sin 2dx = 2dx$

$$\sin(2dx) = 2(\sin dx)(\cos dx)$$

$$\boxed{\sin dx = dx}$$

$$\boxed{\cos dx = 1}$$

$$= (2\,dx)(1)$$

$$= 2\,dx$$

Memorize

$$\frac{d}{dx}\left(x^r\right) = r\,x^{r-1} \quad \text{for any rational number } r$$

Do you know the story about Gauss as a child?

being asked to sum up the first 100 counting numbers?

Absolutely, I love that story. So, Carl Friedrich Gauss, when he was just a kid, was given this problem by his teacher. The teacher thought it would keep the class busy for a while, you know, adding the numbers 1 through 100. But Gauss recognized a pattern, and he quickly realized he could pair the numbers—1 with 100, 2 with 99, and so on—so each pair summed to 101. He just multiplied and got the total super fast. Pretty clever for a kid, right? So hey, did you ever pull any smart tricks like that in school?

$$x = 1 + 2 + 3 + \cdots + 98 + 99 + 100$$
$$x = 100 + 99 + 98 + \cdots + 3 + 2 + 1$$

$$2x = \underbrace{101 + 101 + \cdots + 101}_{100 \text{ terms}}$$

$$x = \frac{100(101)}{2} = 50(101) = \boxed{5050}$$

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

$$\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}}$$ supplied

Prove by math induction

basis step

let $n=1$ $\quad \sum_{i=1}^{1} i \overset{?}{=} \frac{1(1+1)}{2}$

$\overset{?}{}\ 1(2)$

let $n=1$ $\sum_{i=1}^{n} \ldots \frac{\ldots}{2}$

$$1 \overset{?}{=} \frac{1(2)}{2}$$

$$1 = 1 \checkmark$$

True for $n=1$

Inductive hypothesis

Assume $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for any fixed integer $n \geq 1$

Prove $\sum_{i=1}^{n+1} i = \frac{(n+1)((n+1)+1)}{2} = \frac{(n+1)(n+2)}{2}$

$$\sum_{i=1}^{n+1} i = \left(\sum_{i=1}^{n} i\right) + (n+1)$$

$$= \frac{n(n+1)}{2} + (n+1) \quad \text{Ind hyp.}$$

$$= (n+1)\left(\frac{n}{2} + 1\right)$$

$$= (n+1)\left(\frac{n}{2} + \frac{2}{2}\right)$$

$$= \frac{(n+1)(n+2)}{2} \checkmark$$

1.5

For Exercises 1-18, find the derivative of the given function.

3. $f(x) = \sqrt{1 - 2x}$

$\frac{df}{} = \frac{df}{} \cdot \frac{d}{} \ldots \quad \ldots = f'(g(x)) \cdot g'(x)$

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} \qquad f'(x) = f'(g(x)) \cdot g'(x)$$

$$f(x) = (1-2x)^{\frac{1}{2}}$$

$$f'(x) = \left(\frac{1}{2}\right)(1-2x)^{\frac{1}{2}-1} \frac{d}{dx}(-2x)$$

$$= \left(\frac{1}{2}\right)(1-2x)^{-\frac{1}{2}} (-2)$$

$$\boxed{f'(x) = -(1-2x)^{-\frac{1}{2}}}$$

$$\boxed{f'(x) = -\frac{1}{\sqrt{1-2x}}}$$

$$f(x) = \sqrt{1-2x} = (1-2x)^{\frac{1}{2}}$$

Let $u = 1-2x$

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

$$f(u) = u^{\frac{1}{2}}$$

$$f(u) = u^{\frac{1}{2}}$$

$$\frac{df}{du} = \left(\frac{1}{2}\right)u^{-\frac{1}{2}}$$

$$\frac{du}{dx} = \frac{d}{dx}(1-2x) = -2$$

$$\frac{df}{dx} = \left(\frac{1}{\cancel{2}}\right)u^{-\frac{1}{2}}(-\cancel{2})$$

$$= -u^{-\frac{1}{2}}$$

$$= \boxed{-(1-2x)^{-\frac{1}{2}}}$$

Find derivative

6. $f(x) = \dfrac{\sqrt{x}+1}{\sqrt{x}-1} = \dfrac{x^{\frac{1}{2}}+1}{x^{\frac{1}{2}}-1}$

quotient rule

$$f'(x) = \frac{(x^{\frac{1}{2}}-1)\left(\left(\frac{1}{2}\right)x^{-\frac{1}{2}}\right) - (x^{\frac{1}{2}}+1)\left(\left(\frac{1}{2}\right)(x^{-\frac{1}{2}}\right)}{(x^{\frac{1}{2}}-1)^2}$$

$$\left(x^{\frac{1}{2}} - 1\right)^2$$

$$= \frac{\cancel{\left(\frac{1}{2}\right)} - \frac{x^{-\frac{1}{2}}}{2} - \cancel{\left(\frac{1}{2}\right)} - \left(\frac{1}{2}\right) x^{-\frac{1}{2}}}{x - 2x^{\frac{1}{2}} + 1}$$

$$\boxed{f'(x) = \frac{-x^{-\frac{1}{2}}}{\left(x^{\frac{1}{2}} - 1\right)^2}}$$

Gemini

$$f'(x) = \frac{-1}{\sqrt{x}(\sqrt{x} - 1)^2}$$