01-28-26 MTH 263

1.2 The Derivative: Limit Approach
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1.3 The Derivatives: Infinitesimal Approach
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1.4 Derivatives of Sums, Products and Quotients
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1.3 All definitions and properties of infinitesimals will be supplied.

Memorize

Notation for the derivative of $y = f(x)$: The following are all equivalent:

$$\frac{dy}{dx}\ ,\ f'(x)\ ,\ \frac{d}{dx}(f(x))\ ,\ y'\ ,\ \dot{y}\ ,\ \dot{f}(x)\ ,\ \frac{df}{dx}\ ,\ Df(x)$$

A number δ is an **infinitesimal** if the conditions (a)-(d) hold:
(a) $\delta \neq 0$

(b) if $\delta > 0$ then δ is smaller than any positive real number

(c) if $\delta < 0$ then δ is larger than any negative real number

(d) $\delta^2 = 0$ (and hence all higher powers of δ, such as δ^3 and δ^4, are also 0)

Note: Any infinitesimal multiplied by a nonzero real number is also an infinitesimal, while 0 times an infinitesimal is 0.

Microstraightness Property: For the graph of a differentiable function, any part of the curve with infinitesimal length is a straight line segment.

Let dx be an infinitesimal such that $f(x+dx)$ is defined. Then $dy = f(x+dx) - f(x)$ is also an infinitesimal, and the derivative of $y = f(x)$ at x is the ratio of dy to dx:

$$\frac{dy}{dx} = \frac{f(x+dx) - f(x)}{dx} \tag{1.8}$$

Show that the derivative of $y = f(x) = x^2$ is $\frac{dy}{dx} = 2x$.

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h}$$

$$= \frac{(x+h)^2 - x^2}{h}$$

$$= \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \frac{2xh + h^2}{h} = \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$\boxed{\frac{\Delta f}{\Delta x} = 2x + h}$$

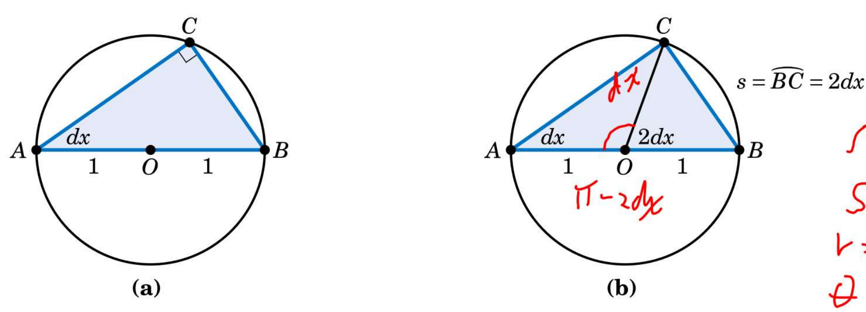
$$f'(x) = \lim_{h \to 0}(2x+h) = 2x + 0 = \boxed{2x}$$

Example 1.4

Show that the derivative of $y = f(x) = x^2$ is $\frac{dy}{dx} = 2x$.

Solution: For any real number x,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{f(x+dx) - f(x)}{dx} \\
&= \frac{(x+dx)^2 - x^2}{dx} \\
&= \frac{\cancel{x^2} + 2x\,dx + (dx)^2 - \cancel{x^2}}{dx} \\
&= \frac{2x\,dx + 0}{dx} \qquad \text{since } dx \text{ is an infinitesimal} \Rightarrow (dx)^2 = 0 \\
&= \frac{2x\,\cancel{dx}}{\cancel{dx}} \\
&= 2x
\end{aligned}$$



s = arc length

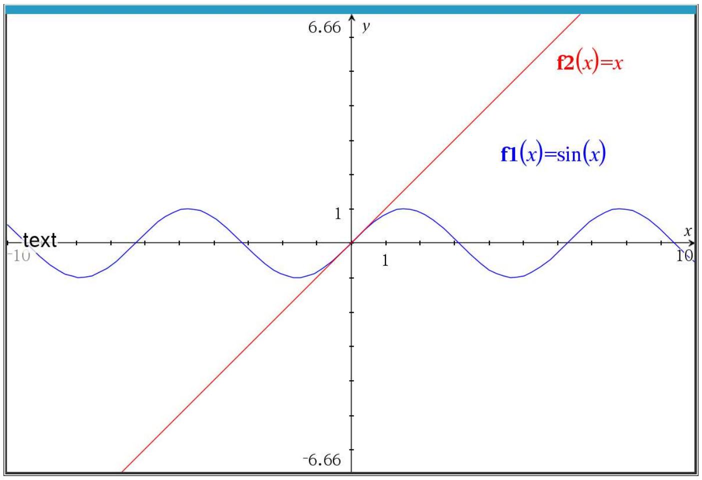
s = rθ

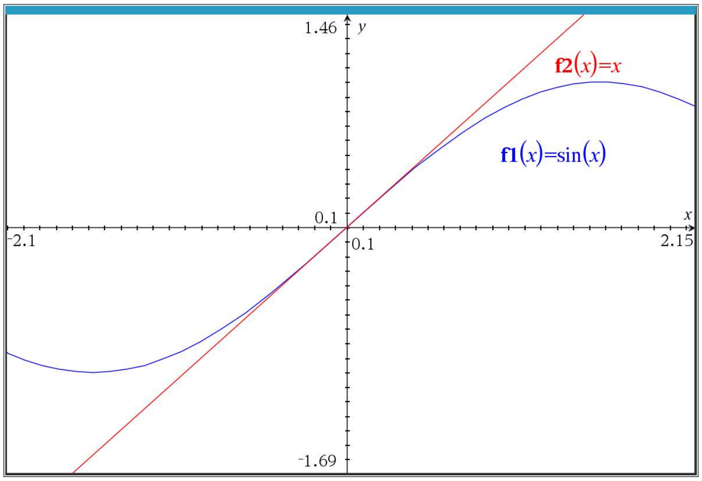
r = radius

θ = central angle

Figure 1.3.2 Circle O: $BC = 2\sin dx$, $\angle BOC = 2\angle BAC$

$\boxed{\sin dx = dx}$





We see from the graph that $\sin(x) \approx x$ for x sufficiently close to zero.

Memorize

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

$f'(x) = \frac{df}{dx}$ With infinitesimals, we can multiply both sides by dx.

$$df = f'(x)\,dx$$

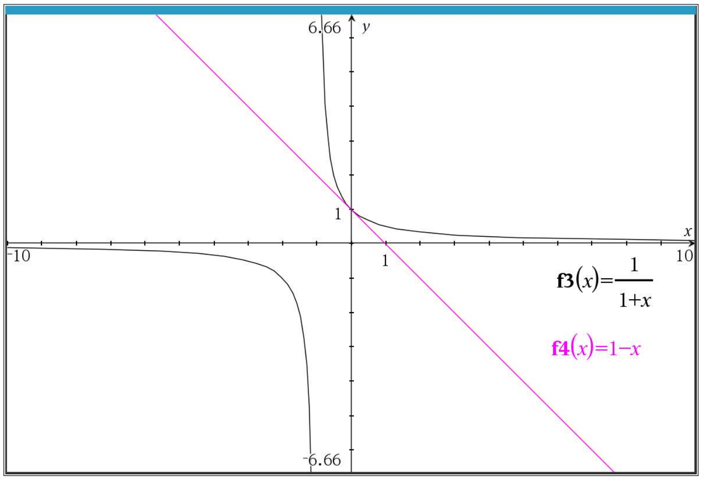
df is called the differential of f.

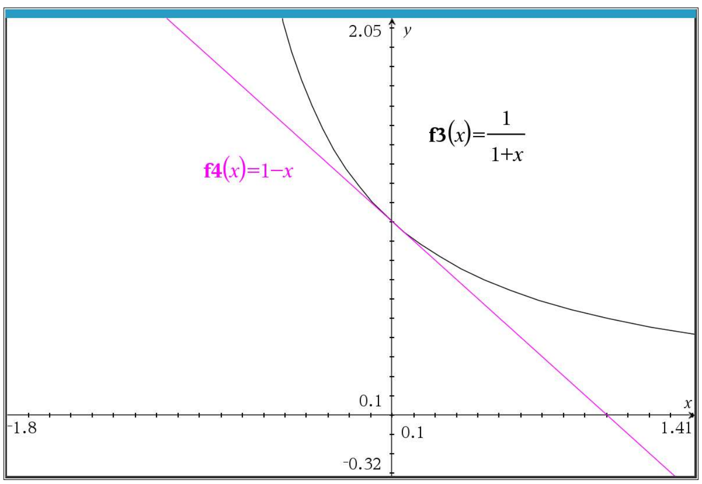
1.2

For Exercises 1-9, let dx be an infinitesimal and prove the given formula.

3. $(dx + 1)^{-1} = 1 - dx$

What approximation formula does this suggest?
$\frac{1}{1+x} \approx 1 - x$ for x close to 0.





3. $(dx + 1)^{-1} = 1 - dx$

$$(dx+1)^{-1} = \frac{1}{dx+1}$$

$$= \frac{1}{1+dx}\left(\frac{1-dx}{1-dx}\right)$$

$$= \frac{1-dx}{1-(dx)^2}$$

$$= \frac{1-dx}{1-0} = \frac{1-dx}{1} = 1-dx \quad \checkmark$$

$$= \frac{1 - ax}{1 - 0} = \frac{1 - ax}{1} = 1 - ax \quad \checkmark$$

1.4
Memorize

Rules for Derivatives: Suppose that f and g are differentiable functions of x. Then:

Sum Rule: $\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$

Difference Rule: $\frac{d}{dx}(f-g) = \frac{df}{dx} - \frac{dg}{dx}$

Constant Multiple Rule: $\frac{d}{dx}(cf) = c \cdot \frac{df}{dx}$ for any constant c

Product Rule: $\frac{d}{dx}(f \cdot g) = f \cdot \frac{dg}{dx} + g \cdot \frac{df}{dx}$

Quotient Rule: $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g \cdot \frac{df}{dx} - f \cdot \frac{dg}{dx}}{g^2}$

Sum Rule: $(f+g)'(x) = f'(x) + g'(x)$

Difference Rule: $(f-g)'(x) = f'(x) - g'(x)$

Constant Multiple Rule: $(cf)'(x) = c \cdot f'(x)$ for any constant c

Product Rule: $(f \cdot g)'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$

Quotient Rule: $\left(\frac{f}{g}\right)'(x) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{(g(x))^2}$

In words, the derivative of a sum is the sum of the derivatives.
The derivative of a difference is the difference of the derivatives.

The proof of the Sum Rule is straightforward. Since $\frac{df}{dx}$ and $\frac{dg}{dx}$ both exist then:

$$\begin{aligned}
\frac{d}{dx}(f+g) &= \frac{(f+g)(x+dx) - (f+g)(x)}{dx} = \frac{f(x+dx) + g(x+dx) - (f(x) + g(x))}{dx} \\
&= \frac{f(x+dx) - f(x) + g(x+dx) - g(x)}{dx} = \frac{f(x+dx) - f(x)}{dx} + \frac{g(x+dx) - g(x)}{dx} \\
&= \frac{df}{dx} + \frac{dg}{dx} \quad \checkmark
\end{aligned}$$

limit approach

$$\frac{d}{dx}(f+g) = \lim_{h \to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h}$$

$$\frac{\Delta (f+g)}{\Delta x} = \frac{f(x+h) + g(x+h) - f(x) - g(x)}{\quad}$$

$$\frac{\Delta(f+g)}{\Delta x} = \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}$$

$$\boxed{\frac{\Delta(f+g)}{\Delta x} = \frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}}$$

$$\frac{d(f+g)}{dx} = \lim_{h\to 0} \frac{\Delta(f+g)}{\Delta x}$$

$$= \lim_{h\to 0} \left(\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h} \right)$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= \frac{df}{dx} + \frac{dg}{dx} \quad \checkmark$$

Standard derivation of the product rule

Prove $\frac{d}{dx}\left((f\cdot g)(x)\right) = g\frac{df}{dx} + f\frac{dg}{dx}$

$$\frac{d}{dx}\left((f\cdot g)(x)\right) = \lim_{h\to 0} \frac{\Delta(f\cdot g)}{\Delta x}$$

$$\frac{\Delta(f\cdot g)}{\Delta x} = \frac{(f\cdot g)(x+h)-(f\cdot g)(x)}{h}$$

$$= \frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

$$\frac{\qquad\qquad}{h}$$

$$= \frac{f(x+h)g(x+h) + f(x)g(x+h) - f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \frac{f(x)g(x+h) - f(x)g(x)}{h}$$

$$\frac{\Delta(fg)}{\Delta x} = g(x+h)\left(\frac{f(x+h)-f(x)}{h}\right) + f(x)\left(\frac{g(x+h)-g(x)}{h}\right)$$

$$\frac{d(fg)}{dx} = \lim_{h\to 0} g(x+h)\left(\frac{f(x+h)-f(x)}{h}\right) + \lim_{h\to 0} f(x)\left(\frac{g(x+h)-g(x)}{h}\right)$$

$$= \lim_{h\to 0} g(x+h)\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} f(x)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

g is diff

$\Rightarrow g$ is continuous

$\Rightarrow \lim_{h\to 0} g(x+h) = g(x)$

$$g(x)\frac{df}{dx} + f(x)\frac{dg}{dx} \quad \checkmark$$