

3.5 Related Rates

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3.6 Differentials

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4 Applications of Derivatives

4.1 Optimization

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3.6 memorize

For a differentiable function $f(x)$, the **differential** of $f(x)$ is

$$df = f'(x) dx \quad (3.7)$$

where dx is an infinitesimal change in x .

Memorize

Let f and g be differentiable functions, and let c be a constant. Then:

(a) $d(c) = 0$

(b) $d(cf) = c df$ (Constant Multiple Rule)

(c) $d(f + g) = df + dg$ (Sum Rule)

(d) $d(f - g) = df - dg$ (Difference Rule)

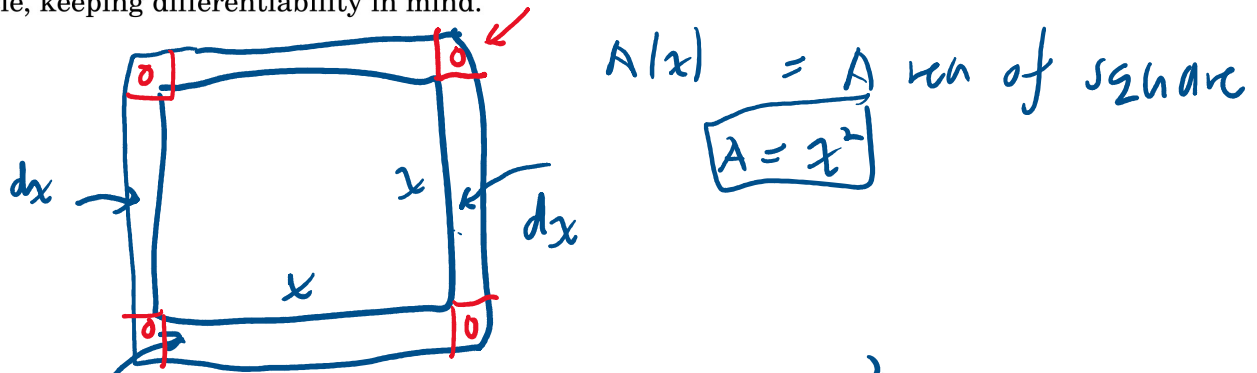
(e) $d(fg) = f dg + g df$ (Product Rule)

(f) $d\left(\frac{f}{g}\right) = \frac{g df - f dg}{g^2}$ (Quotient Rule)

(g) $d(f^n) = n f^{n-1} df$ (Power Rule)

(h) $d(f(g)) = \frac{df}{dg} dg$ (Chain Rule)

The above example answers the question of whether it is a happy coincidence that the derivative of a circle's area turns out to be the circle's circumference—no, it is not! Some other such cases (e.g. the derivative of a sphere's volume is its surface area) are left to the exercises. Note that a similar “coincidence” does *not* occur for a square: if x is the length of each side then the area is x^2 , but the derivative of x^2 is $2x$, which is *not* the perimeter of the square (i.e. $4x$). Why does this not follow the same pattern as the circle? Think about a key difference in the shape of a square in comparison to a circle, keeping differentiability in mind.



$$dA \quad A(x + 2 dx) = (x + 2 dx)^2$$

$$dA = A(x + 2 dx) - A(x)$$

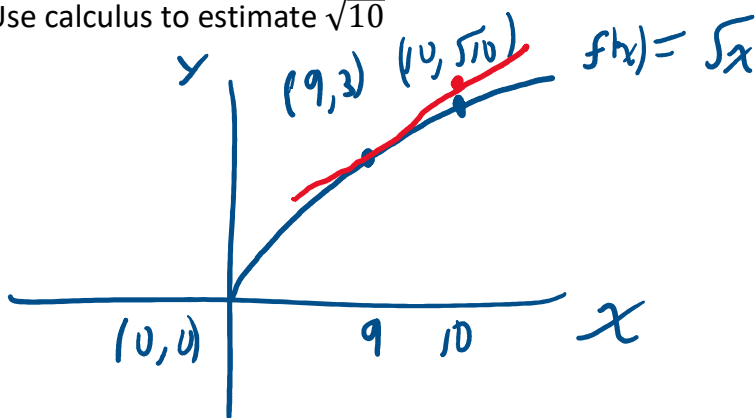
$$= (x + 2 dx)^2 - x^2$$

$$= \cancel{x^2} + 4x dx + 4(\underbrace{dx^2}_{=0}) - \cancel{x^2}$$

$$dA = 4x dx$$

$$\frac{dA}{dx} = 4x \neq 2x = \frac{d}{dx}(x^2)$$

Use calculus to estimate $\sqrt{10}$



$$\text{tan line } y - y_0 = f'(x_0)(x - x_0)$$

$$y - 3 = \frac{d}{dx}(x^{\frac{1}{2}}) \Big|_{x=9} (x - 9)$$

$$y - 3 = \left(\frac{1}{2}\right)(x^{-\frac{1}{2}}) \Big|_{x=9} (x - 9)$$

$$y - 3 = \frac{1}{2\sqrt{x}} \Big|_{x=9} (x - 9)$$

$$y = \frac{1}{2\sqrt{9}}(x - 9) + 3$$

$$y = \frac{1}{6}(x - 9) + 3$$

$$y = \frac{x}{6} - \frac{9}{6} + 3$$

$$\boxed{y = \frac{x}{6} + \frac{3}{2}}$$
 eq of tan line

$$y(10) = \frac{10}{6} + \frac{3}{2} = \frac{5}{3} + \frac{3}{2} = \frac{10}{6} + \frac{9}{6} = \frac{19}{6}$$

$$19/6 = 3.1667 \approx 3.17$$

$$\text{Sqrt}(10) = 3.16227766016838 \approx 3.16$$

Guichard

DEFINITION 6.4.3 Let $y = f(x)$ be a differentiable function. We define a new independent variable dx , and a new dependent variable $dy = f'(x) dx$. Notice that dy is a function both of x (since $f'(x)$ is a function of x) and of dx . We say that dx and dy are **differentials**. \square

Let $\Delta x = x - a$ and $\Delta y = f(x) - f(a)$. If x is near a then Δx is small. If we set $dx = \Delta x$ then

$$dy = f'(a) dx \approx \frac{\Delta y}{\Delta x} \Delta x = \Delta y.$$

Thus, dy can be used to approximate Δy , the actual change in the function f between a and x . This is exactly the approximation given by the tangent line:

$$dy = f'(a)(x - a) = f'(a)(x - a) + f(a) - f(a) = L(x) - f(a).$$

While $L(x)$ approximates $f(x)$, dy approximates how $f(x)$ has changed from $f(a)$. Figure 6.4.2 illustrates the relationships.

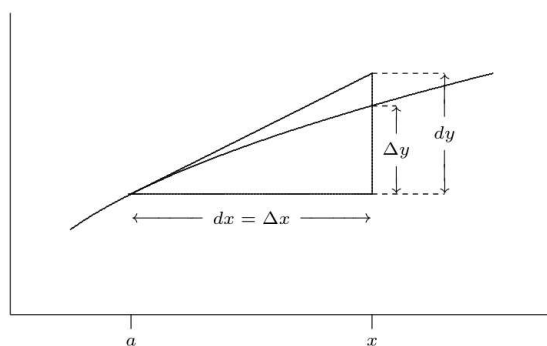


Figure 6.4.2 Differentials.

Find the differential for

$$f(x) = 3x^5 - 2x^2$$

$$\frac{df}{dx} = 15x^4 - 4x$$

$$df = (15x^4 - 4x) dx$$

4.1

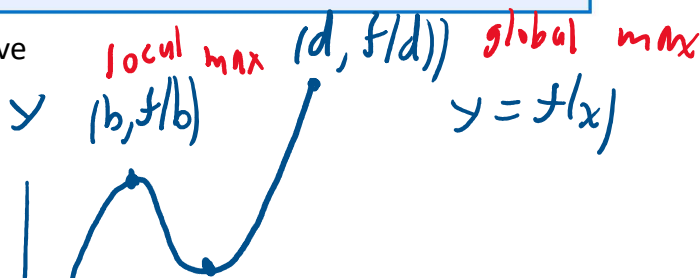
Memorize

A function f has a **global maximum** at $x = c$ if $f(c) \geq f(x)$ for all x in the domain of f . Similarly, f has a **global minimum** at $x = c$ if $f(c) \leq f(x)$ for all x in the domain of f . Say that f has a **local maximum** at $x = c$ if $f(c) \geq f(x)$ for all x “near” c , i.e. for all x such that $|x - c| < \delta$ for some number $\delta > 0$. Likewise, f has a **local minimum** at $x = c$ if $f(c) \leq f(x)$ for all x such that $|x - c| < \delta$ for some number $\delta > 0$.

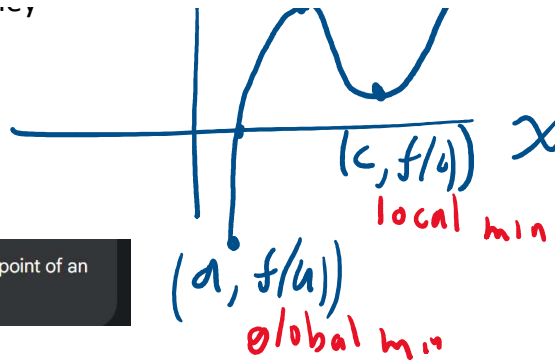
Vocabulary : global = absolute, local = relative

Local max = top of a small hill

Local min = bottom of a small valley



Local min = bottom of a small valley



Google gemini

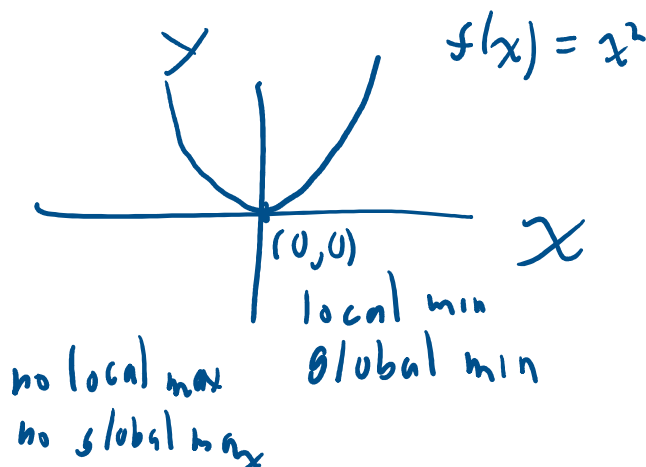
Can a local minimum of a function occur at an endpoint of an interval on which the function is defined?

◆ The answer to whether a local minimum can occur at an endpoint of an interval depends on the precise definition being used. Here's a breakdown:

- **Understanding Local Extrema:**
 - A "local minimum" of a function at a point means that the function's value at that point is less than or equal to the function's values at all nearby points.
 - The concept of "nearby points" is where the nuance arises, especially concerning endpoints.
- **Endpoints and "Nearby Points":**
 - If we strictly define "nearby points" to require an open interval around the point, then endpoints cannot be local minima or maxima. This is because an endpoint does not have points on both sides of it within the function's domain.
 - However, in many contexts, especially when dealing with functions on closed intervals, it's useful to extend the definition of local extrema to include endpoints. In this case, an endpoint can be a local minimum (or maximum) if the function's value there is less than or equal to the function's values at all other points within the function's domain that are "near" it (i.e. on the correct side of the endpoint).

- **Practical Application:**
 - When finding absolute extrema (the overall maximum and minimum values of a function on an interval), endpoints are always considered as potential locations for these extrema.
 - Therefore, while the pure definition of local extrema can exclude end points, in practical application, end points are very important when looking for minimum and maximum values.

In summary, while there is some variation in the strict definition, in practical terms, especially when finding absolute minimum and maximum values, endpoints must be considered.



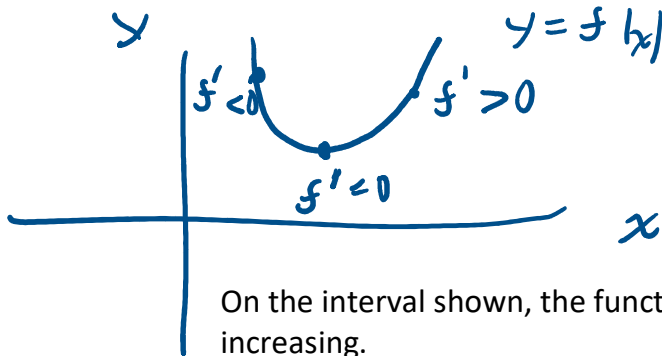
Memorize

Second Derivative Test: Let $x = c$ be a critical point of f (i.e. $f'(c) = 0$). Then:

(a) If $f''(c) > 0$ then f has a local minimum at $x = c$.

(b) If $f''(c) < 0$ then f has a local maximum at $x = c$.

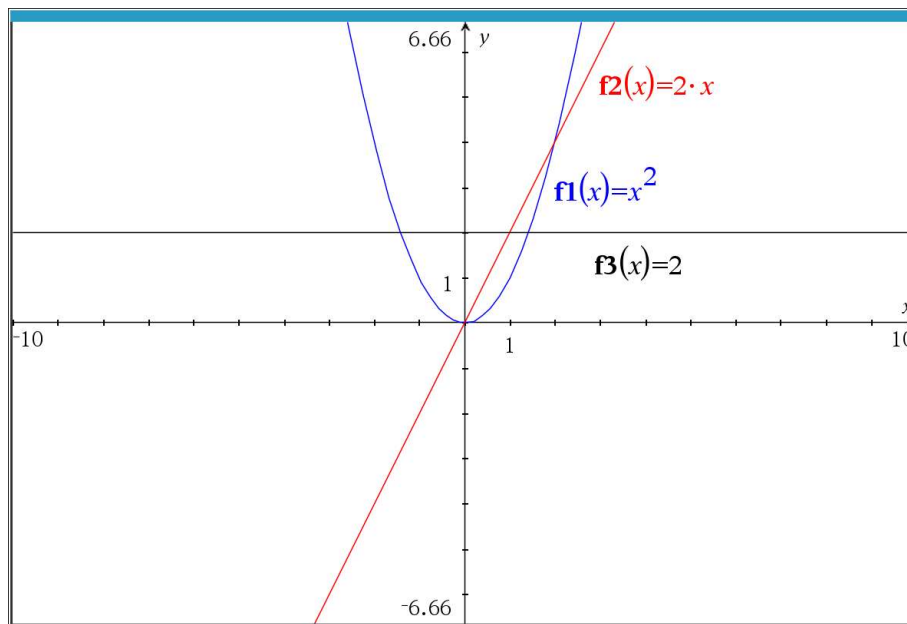
(c) If $f''(c) = 0$ then the test fails.



On the interval shown, the function $f'(x)$ is increasing.

This implies that $f''(x) > 0$

Local min at the critical point, i.e. where $f'(x) = 0$



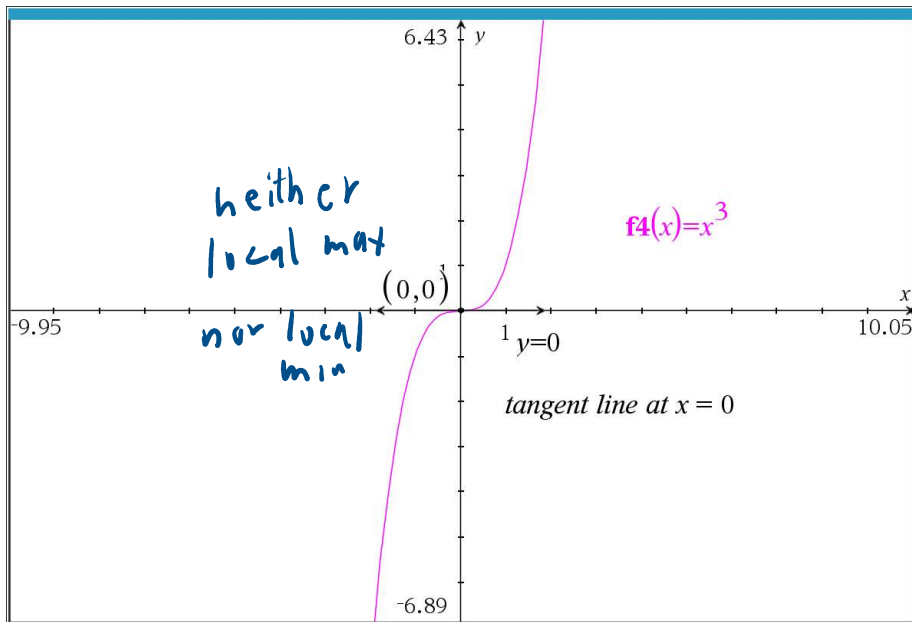
$$f(x) = x^3$$

$$f'(x) = 3x^2 = 0$$

$$\Rightarrow \boxed{x = 0} \text{ critical point}$$

$$f''(x) = 6x$$

$$f''(0) = 0 \text{ test fails}$$

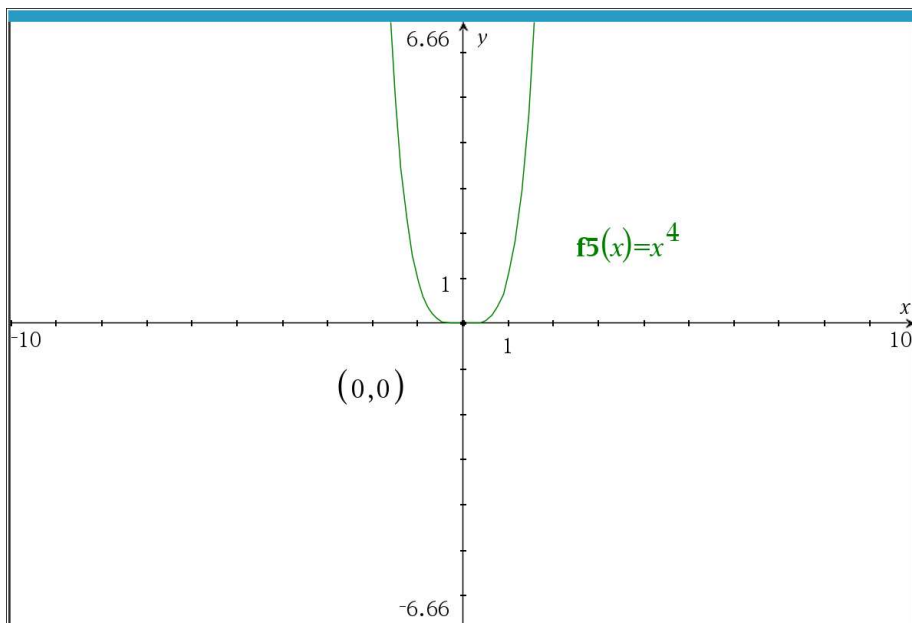


$$f(x) = x^4$$

$$f'(x) = 4x^3 = 0 \Rightarrow \boxed{x=0} \text{ critical point}$$

$$f''(x) = 12x^2 = 0 \text{ test fails}$$

$$x=0$$



It looks like a local and global min at $x = 0$.

How to find a global maximum or minimum

Suppose that f is defined on an interval I . There are two cases:

- 1. The interval I is closed:** The global maximum of f will occur either at an interior local maximum or at one of the endpoints of I whichever of these points provides the largest value of f will be where the global maximum occurs. Similarly, the global minimum of f will occur either at an interior local minimum or at one of the endpoints of I ; whichever of these points provides the smallest value of f will be where the global minimum occurs.
- 2. The interval I is not closed and has only one critical point:** If the only critical point is a local maximum then it is a global maximum. If the only critical point is a local minimum then it is a global minimum.

↑ think parabola