3.3 Continuity
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3.4 Implicit Differentiation
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3.5 Related Rates
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Before class notes
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A

For Exercises 1-9, use implicit differentiation to find $\frac{dy}{dx}$.

7. $\cos(xy) = \sin(x^2y^2)$

$$\frac{d}{dx}\left(\cos(xy)\right) = \frac{d}{dx}\left(\sin(x^2y^2)\right)$$

Remember $y(x)$

$$-\sin(xy)\frac{d}{dx}(xy) = \cos(x^2y^2)\frac{d}{dx}(x^2y^2)$$

$$-\sin(xy)\left(x\frac{dy}{dx} + y\frac{d}{dx}(x)\right) = \cos(x^2y^2)\left(x^2\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x^2)\right)$$

$$-\sin(xy)\left(x\frac{dy}{dx} + y\right) = \cos(x^2y^2)\left(x^2\left(2y\frac{dy}{dx}\right)\right) + 2xy^2\cos(x^2y^2)$$

solve algebraically for $\frac{dy}{dx}$

$$\frac{dy}{dx}\left(-x\sin(xy)\right) - y\sin(xy) = \frac{dy}{dx}\left(2x^2y\cos(x^2y^2)\right) + 2xy^2\cos(x^2y^2)$$

$$\frac{dy}{dx}\left(2x^2y\cos(x^2y^2) + x\sin(xy)\right) = -y\sin(xy) - 2xy^2\cos(x^2y^2)$$

$$\frac{dy}{dx} = \frac{-y\sin(xy) - 2xy^2\cos(x^2y^2)}{2x^2y\cos(x^2y^2) + x\sin(xy)}$$

$$\frac{dy}{dx} = \frac{-\left(2xy^2\cos(x^2y^2) + y\sin(xy)\right)}{2x^2y\cos(x^2y^2) + x\sin(xy)}$$

$$\frac{dy}{dx} = \frac{-y\left(2xy\cos(x^2y^2) + \sin(xy)\right)}{x\left(2xy\cos(x^2y^2) + \sin(xy)\right)}$$

$$\frac{dy}{dx} = \frac{-y\left(2xy\cos(x^2y^2) + \sin(xy)\right)}{x\left(2xy\cos(x^2y^2) + \sin(xy)\right)}$$

$$\boxed{\frac{dy}{dx} = \frac{-y}{x}}$$

7. $-\frac{y}{x}$

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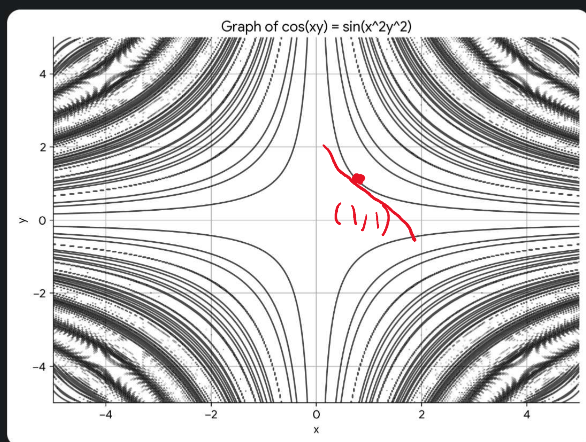
Graph the relation cos(xy) = sin(x^2*y^2)

Analysis

I'll generate a plot to visualize the relation cos(xy) = sin(x^2 * y^2).



Here is the graph of the relation cos(xy) = sin(x^2y^2).

slope of tangent line at (1,1) $= -1 = -\frac{1}{1}$

Hi, can you solve the relation cosine of (x times y) = sine of (x ^2 + y^2) for y in terms of x?

Unfortunately, finding an explicit algebraic solution for "y in terms of x" for the equation cos(xy) = sin($x^2 + y^2$) is extremely difficult, if not impossible. This type of equation is a transcendental equation, and it involves trigonometric functions with products and sums of variables, making it generally unsolvable by standard algebraic methods.
Here's a breakdown of why and what we can do:

- **Complexity:**
  - The equation mixes trigonometric functions (cosine and sine) with polynomial terms (xy and $x^2 + y^2$).
  - The arguments of the trigonometric functions involve both multiplication and addition of the variables, which creates complex dependencies.
  - There are no standard algebraic manipulations to isolate 'y' in this scenario.
- **Implicit Relation:**
  - The equation defines an implicit relation between 'x' and 'y'. This means that 'x' and 'y' are related, but we can't express one explicitly as a function of the other.
- **What we can do:**
  - **Graphical Representation:** As demonstrated previously, we can graph the relation to

visualize the relationship between 'x' and 'y'. This provides valuable insight into the solution set.

- **Numerical Methods:** We can use numerical methods (e.g., Newton-Raphson method) to find approximate solutions for specific values of 'x'. However, this won't give us a general formula for 'y' in terms of 'x'.
- **Analysis:** we can analyse certain aspects of the equation. For example we can analyse the range of possible solutions, and look for points of symmetry.

In summary, while we cannot provide an explicit formula for 'y' in terms of 'x', we can visualize and analyze the relationship using graphical and numerical methods.

3.3

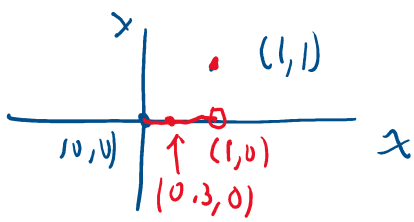
For Exercises 1-18, indicate whether the given function $f(x)$ is continuous or discontinuous at the given value $x = a$ by comparing $f(a)$ with $\lim_{x \to a} f(x)$.

4. $f(x) = \lfloor x \rfloor$; at $x = 0.3$

$\lfloor x \rfloor$ = greatest integer $\leq x$

$\lfloor 5 \rfloor = 5$

$\lfloor 3.7 \rfloor = 3$

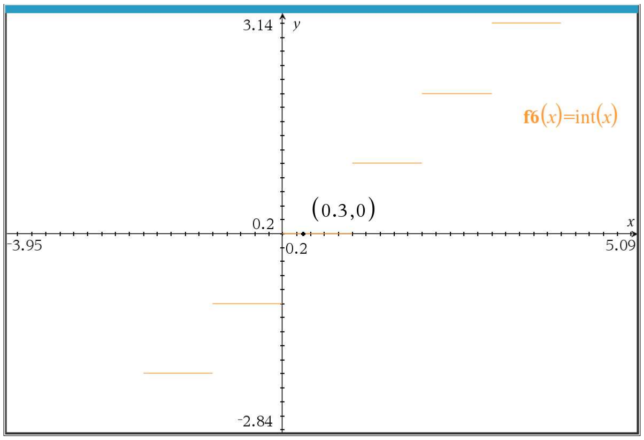


$$\lim_{x \to 0.3} \lfloor x \rfloor = \lfloor 0.3 \rfloor$$

$$\lim_{x \to 0.3} (0) = 0$$

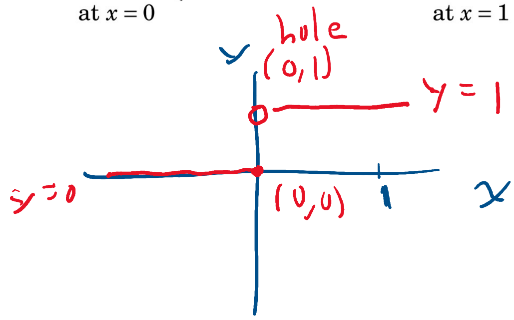
$$0 = 0 \checkmark$$

$\therefore \lfloor x \rfloor$ is cont. at $x = 0.3$



10. $f(x) = \begin{cases} 0 & \text{if } x \le 0, \\ 1 & \text{if } x > 0; \end{cases}$ at $x = 0$

11. $f(x) = \begin{cases} 0 & \text{if } x \le 0, \\ 1 & \text{if } x > 0; \end{cases}$ at $x = 1$



$\lim_{x \to 0^-} f(x) = 0$

$\lim_{x \to 0^+} f(x) = 1$

$0 \neq 1$

$\therefore$ f is not cont at $x = 0$

$\lim_{x \to 1^-} f(x) = 1$

$\lim_{x \to 1^+} f(x) = 1$

$\therefore \lim_{x \to 1} f(x) = 1$

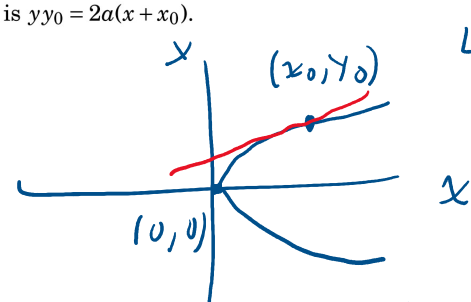
$f(1) = 1$

$\therefore \lim_{x \to 1} f(x) = f(1)$

3.4: 14

14. Show that at every point (x_0, y_0) on the curve $y^2 = 4ax$, the equation of the tangent line to the curve is $yy_0 = 2a(x + x_0)$.

Let $a > 0$



generic sketch

$y - y_0 = m(x - x_0)$ point-slope equation of tangent line

$m = \frac{dy}{dx}(x_0, y_0)$

$$m = \frac{dy}{dx}(x_0, y_0)$$

$$y^2 = 4ax$$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(4ax) = 4a\frac{dx}{dx} = (4a)(1) = 4a$$

$$(2y)\frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{4a}{2y}$$

$$\boxed{\frac{dy}{dx} = \frac{2a}{y}}$$

$$\boxed{\frac{dy}{dx}(x_0, y_0) = \frac{2a}{y_0}}$$

$$y - y_0 = \frac{2a}{y_0}(x - x_0)$$

$$yy_0 - y_0^2 = 2a(x - x_0)$$

$$\boxed{yy_0 = 2a(x - x_0) + y_0^2}$$

$$y^2 = 4ax$$

$$\Rightarrow y_0^2 = 4a\,x_0$$

$$yy_0 = 2a(x - x_0) + 4a\,x_0$$

$$yy_0 = 2a(x - x_0) + 4a x_0$$

$$yy_0 = 2ax - 2a x_0 + 4a x_0$$

$$yy_0 = 2ax + 2a x_0$$

$$\boxed{yy_0 = 2a(x + x_0)}$$

3.5

Google Gemini

What are related rate problems?

Related rates problems are a type of calculus problem that involves finding the rate at which a quantity changes by relating it to other quantities whose rates of change are known.
Here's a breakdown:

Core Concept:

- These problems deal with situations where several variables are changing over time, and these variables are related to each other.
- The goal is to determine how the rate of change of one variable affects the rate of change of another.
- Essentially, it's about understanding how the rates of different quantities are "related."

The independent variable does not have to be time.

Key Elements:

- **Variables:** Quantities that change with respect to time (e.g., distance, volume, area, angle).
- **Rates of Change:** Derivatives of these variables with respect to time (e.g., dx/dt, dv/dt, dA/dt).
- **Relationships:** Equations that connect the variables (often from geometry, trigonometry, or other formulas).
- **Implicit Differentiation:** A calculus technique used to differentiate equations where variables are related, especially when they are functions of time.

Common Scenarios:

- **Moving Objects:** How the distance between two moving objects changes over time.
- **Filling Containers:** How the height or volume of liquid in a container changes as it's being filled.
- **Changing Shapes:** How the area or volume of a shape changes as its dimensions change.
- **Shadows:** How the length of a shadow changes as an object moves.

General Steps to Solve Related Rates Problems:

1. **Identify Variables and Rates:** Determine the quantities that are changing and their rates of

change.

2. **Establish a Relationship:** Find an equation that relates the variables.
3. **Differentiate with Respect to Time:** Use implicit differentiation to find an equation that relates the rates of change.
4. **Substitute Known Values:** Plug in the given rates and values at the specific instant.
5. **Solve for the Unknown Rate:** Solve the resulting equation for the rate you need to find.

In essence, related rates problems allow us to use the principles of calculus to analyze dynamic situations and understand how changes in one quantity affect changes in another.

Copilot

What are related rate problems?

Related rate problems are all about how different rates change in relation to each other. For example, if a balloon is being inflated, you might look at how the volume and the radius of the balloon change over time. It's a big hit in calculus!

3.5

2. The radius of a sphere decreases at a rate of 3 mm/hr. Determine how fast the volume and surface area of the sphere are changing when the radius is 5 mm.

supplied Let V = volume of sphere
r = radius of sphere
A = surface area of sphere

$V = \frac{4\pi r^3}{3}$ $A = 4\pi r^2$

Let t = time

$\frac{dr}{dt} = -3 \frac{mm}{hr}$

Find $\frac{dV}{dt}$, $\frac{dA}{dt}$, when $r = 5$ mm

$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4\pi}{3} r^3\right)$

$= \frac{4\pi}{3} \frac{d}{dt}(r^3)$

$= \left(\frac{4\pi}{3}\right)(3r^2)\left(\frac{dr}{dt}\right)$

$$\frac{4\pi}{3}\frac{d}{dt}(r^3)$$

$$=\left(\frac{4\pi}{3}\right)(3r^2)\left(\frac{dr}{dt}\right)$$

$$=(4\pi)(5\text{ mm})^2\left(-3\ \frac{\text{mm}}{\text{hr}}\right)$$

$$\boxed{\frac{dV}{dt}=-300\pi\ \frac{\text{mm}^3}{\text{hr}}}$$

$$\frac{dA}{dt}=\frac{d}{dt}(4\pi r^2)$$

$$=4\pi\frac{d}{dt}(r^2)$$

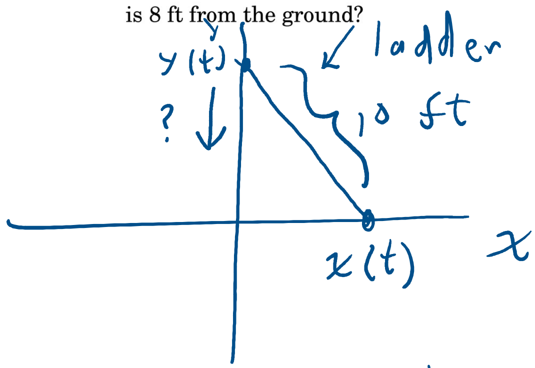
$$=4\pi(2r)\frac{dr}{dt}$$

$$=4\pi((2)(5\text{ mm}))\left(-3\ \frac{\text{mm}}{\text{hr}}\right)$$

$$\boxed{\frac{dA}{dt}=-120\pi\ \frac{\text{mm}^2}{\text{hr}}}$$

3.5:

4. A 10-ft ladder is leaning against a wall on level ground. If the bottom of the ladder is dragged away from the wall at the rate of 5 ft/s, how fast will the top of the ladder descend at the instant when it is 8 ft from the ground?



Let $x(t)$ = distance from the wall to the bottom of the ladder

Let $x(t)$ = distance from the wall to the bottom of the ladder at time t

Let $y(t)$ = distance from the ground to the top of the ladder

$$\frac{dx}{dt} = 5\ \frac{ft}{s}$$

Find $\frac{dy}{dt}$ when $y = 8$ ft

Pyth. Thm $\Rightarrow x^2 + y^2 = 10^2 = 100$

$$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(100)$$

$$(2x)\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$(2x)\left(5\ \frac{ft}{sec}\right) + 2y\frac{dy}{dt} = 0$$

$$(2x)\left(5\ \frac{ft}{sec}\right) + 2(8ft)\frac{dy}{dt} = 0$$

$$x^2 + y^2 = 100\ ft^2$$

$$x^2 = 100\ ft^2 - y^2$$

$$x^2 = (100 - 64)\ ft^2$$

$$x^2 = 36\ ft^2$$

$$\boxed{x = 6\ ft}$$

$$2(6\ ft)(5)\left(\frac{ft}{sec}\right) + (16\ ft)\frac{dy}{dt} = 0$$

$$60\ \frac{ft^2}{sec} + (16\ ft)\frac{dy}{dt} = 0$$

$$60\,\frac{ft}{sec} + (16\,ft)\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{60\,\frac{ft^2}{sec}}{16\,ft}$$

$$\boxed{\frac{dy}{dt} = -\frac{15}{4}\,\frac{ft}{sec}}$$

Your name MTH 263 bonus quiz 2

1. Let $f(x) = \begin{cases} x^2 \text{ for } x < 0 \\ x \text{ for } x \geq 0 \end{cases}$

Is f(x) continuous at x = 0? Why or why not?

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 = 0^2 = 0$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0 = 0$$

$$\therefore \lim_{x \to 0} f(x) = 0$$

$$f(0) = 0$$

$$\therefore \lim_{x \to 0} f(x) = f(0) = 0$$

$\therefore f(x)$ <u>is</u> continuous at $x = 0$

2. Find $\frac{dy}{dx}$ for $xy + \sin(y) = 1$.

$$\frac{d}{dx}(xy) + \frac{d}{dx}(\sin y) = \frac{d}{dx}(1)$$

$$x\frac{dy}{dx} + y\frac{dx}{dx} + \cos y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x + \cos y) = -y$$

$$\boxed{\frac{dy}{dx} = \frac{-y}{x + \cos y}}$$