03-11-25 MTH 263

Exam 2: 1.6, 2.1 -2.4, 3.1-3.2   omit big O
Thursday, 03/13/25

____

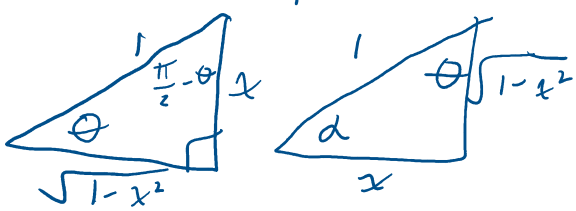
Quiz 4

1. 2.2:

17. Find the derivative of $y = \sin^{-1}x + \cos^{-1}x$. Explain why no derivative formulas were needed.

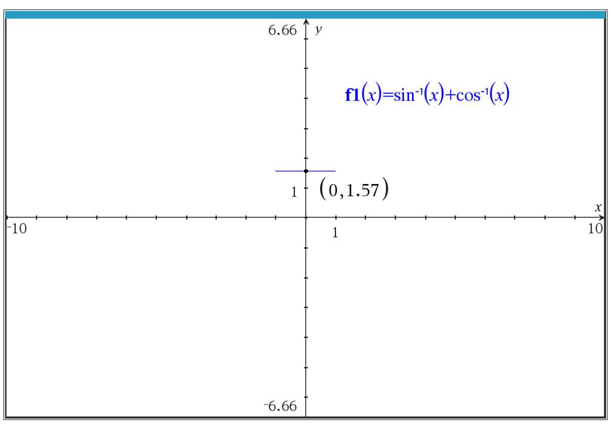
Let $\theta = \sin^{-1}x$
$\alpha = \cos^{-1}x$

$\Rightarrow \sin\theta = x = \frac{x}{1}$
$\cos\alpha = x = \frac{x}{1}$



$$\boxed{\theta = \frac{\pi}{2} - \alpha}$$

$$\frac{d}{dx}(\theta + \alpha)$$
$$= \frac{d}{dx}\left(\frac{\pi}{2} - \alpha + \alpha\right)$$
$$= \frac{d}{dx}\left(\frac{\pi}{2}\right) = \boxed{0}$$



graph looks like
$f(x) = \frac{\pi}{2}$
$-1 \leq x \leq 1$

2. 1.6: 4

For Exercises 1-6 find the second derivative of the given function.

$$f(x) = \frac{\sin x}{x}$$

$$f'(x) = \frac{x\frac{d}{dx}(\sin x) - \sin x\frac{d}{dx}(x)}{x^2}$$

$$f'(x) = \frac{x\cos x - \sin x}{x^2}$$

$$f''(x) = \frac{x^2(-x\sin x + \cos x - \cos x) - (x\cos x - \sin x)(2x)}{x^4}$$

$$f''(x) = -x^3\sin x - 2x^2\cos x + 2x\sin x$$

$$f''(x) = \frac{-x^3 \sin x - 2x^2 \cos x + 2x \sin x}{x^4}$$

$$\boxed{f''(x) = \frac{-x^2 \sin x - 2x \cos x + 2 \sin x}{x^3}}$$

TI-nspire check

$$\frac{d^2}{dx^2}\left(\frac{\sin(x)}{x}\right) \qquad \left(\frac{2}{x^3}-\frac{1}{x}\right)\cdot \sin(x)-\frac{2\cdot \cos(x)}{x^2}$$

3. 2.1

For Exercises 1-8, show that the given function $y = f(x)$ is one-to-one over the given interval, then find the formulas for the inverse function f^{-1} and its derivative. Use Example 2.2 as a guide, including putting f^{-1} and its derivative in terms of x.

4. $f(x) = \sqrt{x}$, for all $x \geq 0$

Assume $\sqrt{c} = \sqrt{d}$

$\Rightarrow c = d$

$\therefore f$ is 1-1

Find $f^{-1}(x)$

$y = \sqrt{x}$

$x = \sqrt{y}$

$x^2 = y$

$\boxed{f^{-1}(x) = x^2}$

$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{d}{dx}\left(x^2\right) = \boxed{2x}$

Using Example 2.2

$y = f(x) = \sqrt{x}$ → rewrite by switching x, y

$\Rightarrow x = f(y) = \sqrt{y}$

$\Rightarrow y = f^{-1}(x) = x^2$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{d}{dy}\left(y^{\frac{1}{2}}\right)}$

$= \frac{1}{\left(\frac{1}{2}\right)\left(y^{-\frac{1}{2}}\right)} = 2\sqrt{y} = \boxed{2x}$

4. Guichard 4.10

8. Find the derivative of $\ln((\arcsin x)^2)$

Let $y = \ln((\arcsin x)^2)$

$$\frac{dy}{dx} = \frac{1}{(\arcsin x)^2}\frac{d}{dx}\left((\arcsin x)^2\right)$$

$$= \frac{1}{(\arcsin x)^2}(2)(\arcsin x)\frac{d}{dx}(\arcsin x)$$

$$= \frac{2}{\arcsin x}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

TI-nspire check

$\frac{d}{dx}\left(\ln\left(\left(\sin^{-1}(x)\right)^2\right)\right)$ $\frac{2}{\sqrt{1-x^2}\cdot\sin^{-1}(x)}$

Warning

Domain of the result might be larger than the domain of the input.

OK

5. Find $\lim_{x\to 3}(6x-10)$.
Verify your limit using the formal $\in -\delta$ definition of limit.

$$\lim_{x\to 3}(6x-10) = 6(3)-10 = 18-10 = \boxed{8}$$

Let $\varepsilon > 0$. Find $\delta > 0$ such that

$$0 < |x-3| < \delta \Rightarrow |(6x-10)-8| < \varepsilon$$

$$|(6x-10)-8| < \varepsilon$$

$$\Leftrightarrow |6x-18| < \varepsilon$$

$$\Leftrightarrow 6|x-3| < \varepsilon$$

$$\Leftrightarrow |x-3| < \frac{\varepsilon}{6}$$

$$\therefore \boxed{\text{Let } \delta = \frac{\varepsilon}{6}}$$

2.1: 7

A

For Exercises 1-8, show that the given function $y = f(x)$ is one-to-one over the given interval, then find the formulas for the inverse function f^{-1} and its derivative. Use Example 2.2 as a guide, including putting f^{-1} and its derivative in terms of x.

7. $f(x) = \frac{1}{x^2}$, for all $x > 0$

Assume $f(c) = f(d)$ with $c > 0$, $d > 0$

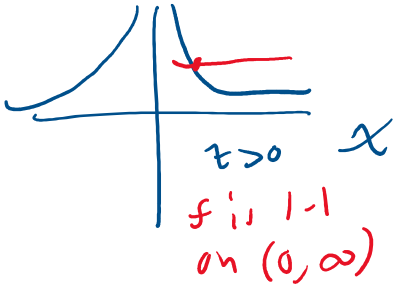
Prove $c = d$

$$\frac{1}{c^2} = \frac{1}{d^2}$$

$$c^2 = d^2$$

$$c = \pm d$$

$$\therefore c = d$$



$y = f(x) = \frac{1}{x^2}, \quad x > 0$

switch x, y

$x = \frac{1}{y^2}$

solve for y

$y^2 = \frac{1}{x}$

$y = \frac{1}{\sqrt{x}}$

$$\boxed{f^{-1}(x) = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}}$$

$$\frac{d}{dx} f^{-1}(x) = \frac{d}{dx}\left(x^{-\frac{1}{2}}\right)$$

$$= \boxed{\left(-\frac{1}{2}\right) x^{-3/2}}$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{d}{dy}\left(y^{-2}\right)}$$

$$\frac{dy}{dx} = \frac{1}{-2y^{-3}} = -\frac{1}{2\left(\frac{1}{\sqrt{x}}\right)^{-3}}$$

$$= -\frac{1}{2\left(\sqrt{x}\right)^{3}} = \boxed{\frac{-1}{2x^{3/2}}}$$

$$= \boxed{-\frac{x^{-3/2}}{2}}$$

$$= \boxed{-\frac{x^{-3/2}}{2}}$$ ✓

Use either method

2,3: 19

19. Suppose it takes 8 hours for 30% of a radioactive substance to decay. Find the half-life of the substance.

$A(t) = y = A_0 e^{kt} \quad (k < 0)$ Supplied

A_0 = initial amount

$$A(0) = A_0 e^{(k)(0)} = A_0 e^0 = (A_0)(1) = A_0$$

$$\therefore A(0) = A_0$$

Definition: the half-life of a radioactive substance is the time for half of the substance to decay, or for half of the substance to remain.

30% decays implies that 70% remains.

$$A(8) = A_0 e^{k(8)} = 0.7 A_0$$

$$\Rightarrow e^{8k} = 0.7$$

$$\Rightarrow \ln(e^{8k}) = \ln(0.7)$$

$$\Rightarrow 8k = \ln(0.7)$$

$$\boxed{k = \frac{\ln(0.7)}{8}}$$

$$A(t) = A_0 e^{\frac{\ln(0.7)}{8} t}$$

Find t such that

$$A(t) = \frac{A_0}{2} = A_0 e^{\frac{\ln(0.7)}{8} t}$$

$$\frac{1}{2} = e^{\left[\frac{\ln(0.7)}{8}\right] t}$$

scratch work

$\frac{1}{2} = e^{\frac{[\ln(0.7)]t}{8}}$

$\frac{1}{2} = e^{\frac{t}{8}\left(\ln(0.7)\right)}$

$\frac{1}{2} = e^{\ln(0.7)^{t/8}}$

$\frac{1}{2} = 0.7^{t/8}$

$\ln\left(\frac{1}{2}\right) = \ln\left(0.7^{t/8}\right)$

$\ln\left(\frac{1}{2}\right) = \frac{t}{8}\ln(0.7)$

$\boxed{t = \frac{8\ln\left(\frac{1}{2}\right)}{\ln(0.7)}} = \frac{\ln\left(\frac{1}{2}\right)}{\frac{\ln(0.7)}{8}}$

$= \frac{\ln(1) - \ln(2)}{\frac{\ln(0.7)}{8}}$

$= \frac{0 - \ln 2}{\frac{\ln(0.7)}{8}}$

$= \boxed{\frac{-\ln 2}{k}}$

scratch work

$\ln(a^r) = r\ln a$

$e^{mn} = (e^m)^n$

$$t_H = -\frac{\ln 2}{k} \quad \text{and} \quad k = -\frac{\ln 2}{t_H}$$

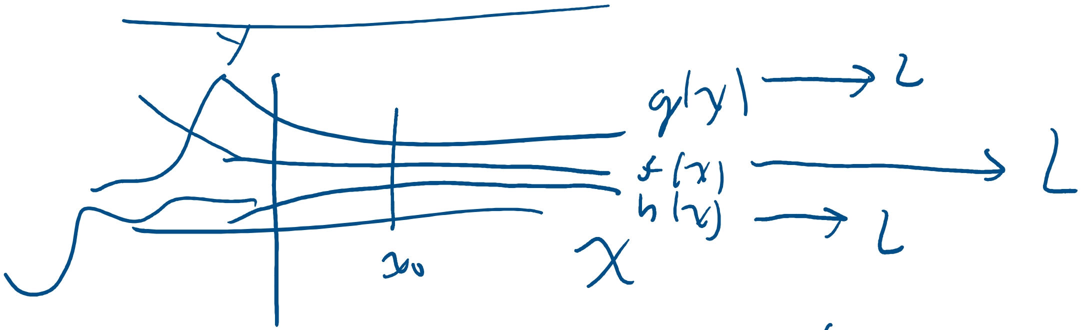
$t \approx 15.55$ hr.

$\frac{8\cdot\ln\left(\frac{1}{2}\right)}{\ln(0.7)}$	15.5469

extra $\left\{\begin{array}{l} A(t) = A_0 e^{kt}, \quad A_0, k \text{ constants} \\ \Rightarrow A'(t) = A_0 k e^{kt} \\ \qquad = k\left(A_0 e^{kt}\right) \\ dA = k\,A(t) \end{array}\right.$

$= k \; (A_0 e \quad)$

$$\frac{dA}{dt} = k\,A(t)$$



$h(x) \le f(x) \le g(x)$ for $x \geq x_0$

Let $y = \arcsin x$

derive the formula for $\frac{dy}{dx}$

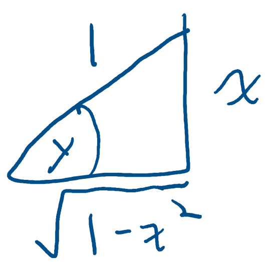
$$y = \sin^{-1}(x)$$

$$\Rightarrow \boxed{\sin y = x}$$

$$\frac{d}{dx}(\sin y) = \frac{dx}{dx}$$

$$(\cos y)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \boxed{\frac{1}{\cos y}} = \boxed{\frac{1}{\sqrt{1-x^2}}}$$



$$\Rightarrow \cos y = \sqrt{1-x^2}$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$= \frac{1}{\frac{d}{dy}(\sin y)}$$

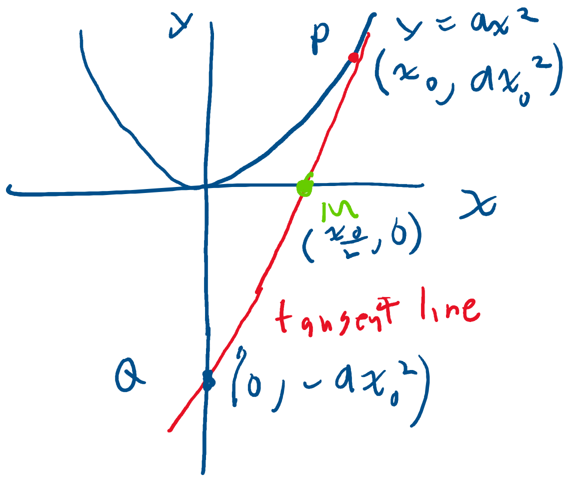
$$= \boxed{\frac{1}{\cos y}}$$

$$= \frac{1}{\sqrt{1-\sin^2 y}}$$

$$= \frac{1}{\sqrt{1-x^2}}$$

3.1: 27

27. For a constant $a > 0$, let P be a point on the curve $y = ax^2$, and let Q be the point where the tangent line to the curve at P intersects the y-axis. Show that the x-axis bisects the line segment $\overline{PQ}$.



Let P be (x_0, ax_0^2)

slope of PQ

$= \frac{d}{dx}(y)$ at P

$= \frac{d}{dx}(ax^2)$

$= 2ax$

$$\boxed{\frac{dy}{dx}(x_0) = 2ax_0}$$

Prove QM = PM

$\frac{dy}{dx}\Big|_{x = x_0}$

$$\boxed{y - ax_0^2 = 2ax_0(x - x_0)}$$

$y = 2ax_0x - 2ax_0^2 + ax_0^2$

$\boxed{y = 2ax_0x - ax_0^2}$ tangent line

$y(0) = (2a_0x_0)(0) - ax_0^2$

$y(0) = -ax_0^2$

Let $2\cancel{a}\cancel{x_0}x - \cancel{a}x_0^{\cancel{2}} = 0$

solve for x

$2x - x_0 = 0$

$$\boxed{x = \frac{x_0}{2}}$$

use distance formula or midpoint finish

3.2

A

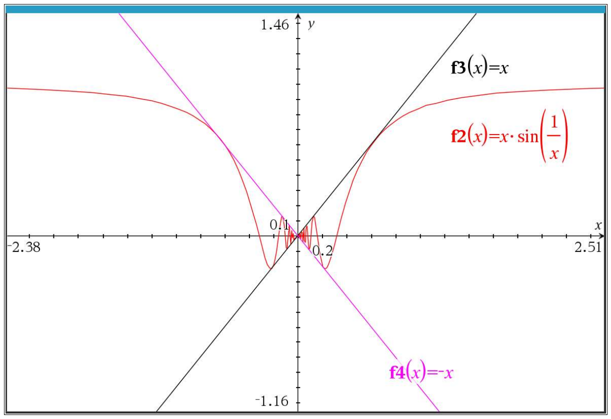
For Exercises 1-18 evaluate the given limit.

12. $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$?

$\downarrow$ 0 $\quad \downarrow$ bounded between ± 1

$\left|x \sin \frac{1}{x}\right| \le |x|\left|\sin \frac{1}{x}\right|$

$$\left|x \sin \tfrac{1}{x}\right| \le |x|\left|\sin \tfrac{1}{x}\right|$$
$$\le |x|(1) = \le |x|$$
$$\lim_{x\to 0} \left|x \sin \tfrac{1}{x}\right| \le \lim_{x\to 0} |x| = 0$$
$$\Rightarrow \lim_{x\to 0} x \sin \tfrac{1}{x} = 0$$



claim $-x \le x \sin x \le x \Leftrightarrow |x \sin x| \le x$

$|x \sin x|$
$= |x|\,|\sin x| \le |x|(1) = |x|$

if so $\lim_{x\to 1} (-x) = 0 = \lim_{x\to 0} x$

Then squeeze Thm $\Rightarrow \lim_{x\to 0} x \sin x = 0$

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$
$$-x \le x \sin\left(\frac{1}{x}\right) \le x$$

After class notes

Example 3.6

Show that $\lim\limits_{x\to a} x = a$ for any real number a.

Solution: Though the limit is obvious, the following "epsilon-delta" proof shows how to use the formal definition. The idea is to let $\epsilon > 0$ be given, then "work backward" from the inequality $|f(x) - L| < \epsilon$ to get an inequality of the form $|x - a| < \delta$, where $\delta > 0$ usually depends on ϵ. In this case the limit is $L = a$ and the function is $f(x) = x$, so since

$$|f(x) - a| < \epsilon \quad \Leftrightarrow \quad |x - a| < \epsilon\,,$$

then choosing $\delta = \epsilon$ means that

$$0 < |x - a| < \delta \quad \Rightarrow \quad |x - a| < \epsilon \quad \Rightarrow \quad |f(x) - a| < \epsilon\,,$$

which by definition means that $\lim\limits_{x\to a} x = a$.

I agree that the limit is obvious.
Verify the limit with the formal definition of limit.

Let $\varepsilon > 0$.

$\lim\limits_{x\to a} f(x) = L$

Let $\varepsilon > 0$

find $\delta > 0$ such that

Verify the limit with the formal definition of limit.

Let $\varepsilon > 0$.

Find $\delta > 0$ such that

$0 < |x - a| < \delta \Rightarrow |x - a| < \varepsilon$

$\boxed{\text{Let } \delta = \varepsilon}$

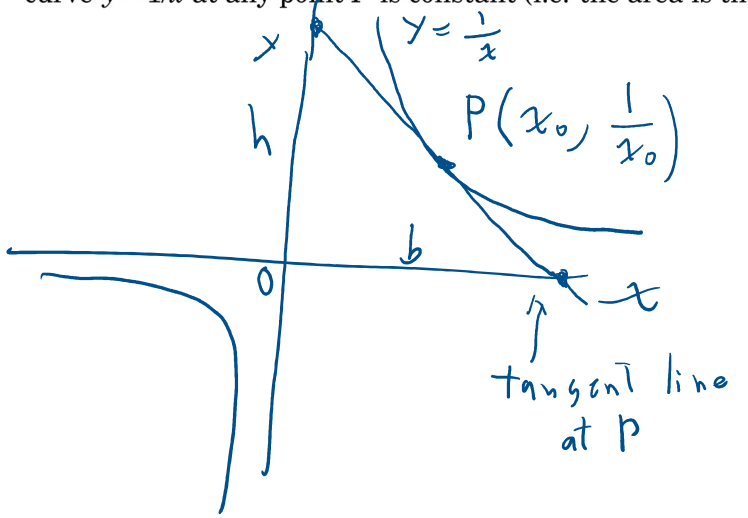
Let $\varepsilon > 0$

find $\delta > 0$ such that

$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

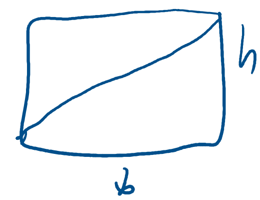
3.1

26. Show that the area of the triangle formed by the x-axis, the y-axis, and the tangent line to the curve $y = 1/x$ at any point P is constant (i.e. the area is the same for all P).



Let A = area of triangle

$A = \frac{(\text{base})(\text{height})}{2}$



$A = \frac{bh}{2}$, b = base

h = height

slope of tangent line $= \frac{dy}{dx}(x_0)$

point of tangency $= (x_0, \frac{1}{x_0})$

2.4: 10

10. Show that for all constants k the function $y = Aa^{\frac{kx}{\ln a}}$ satisfies the differential equation $\frac{dy}{dx} = ky$. Does this contradict the statement made in Section 2.3 that the only solution to that differential equation is of the form $y = Ae^{kx}$? Explain your answer.

· can we write $a^{\frac{kx}{\ln a}}$ in the form e^{kx} ?

This is algebra

2.3

13. Show that $\frac{d}{dx}(\ln(kx)) = \frac{1}{x}$ for all constants $k > 0$.

$\frac{d}{}(\ln x)$

13. Show that $\frac{d}{dx}(\ln(kx)) = \frac{1}{x}$ for all constants $k > 0$.

$$\frac{d}{dx}(\ln(kx)) = \frac{1}{kx}\frac{d}{dx}(kx)$$
$$= \frac{1}{kx}(k)$$
$$= \frac{1}{x}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}\ln(u(x)) = \frac{1}{u(x)}\frac{du}{dx}$$

$$\frac{d}{dx}(\ln(kx))$$
$$= \frac{d}{dx}(\ln k + \ln x)$$
$$= \frac{d}{dx}(\ln k) + \frac{d}{dx}(\ln x)$$
$$= 0 + \frac{1}{x}$$
$$= \boxed{\frac{1}{x}}$$

2.3 **21.** If a certain cell population grows exponentially—i.e. is of the form $A_0 e^{kt}$ with $k > 0$—and if the population doubles in 6 hours, how long would it take for the population to quadruple?

Let $A(t)$ = number of cells at time t

$A(t) = A_0 e^{kt}$, A_0 = original population, $k > 0$

$$A(6) = 2A_0$$
$$A(6) = A_0 e^{6k}$$
$$2A_0 = A_0 e^{6k}$$
$$2 = e^{6k}$$
$$\ln(2) = \ln(e^{6k})$$
$$\ln(2) = 6k$$
$$k = \frac{\ln(2)}{6}$$

$$\boxed{k = \frac{\ln(2)}{6}}$$

$$\boxed{A(t) = A_0 e^{\frac{\ln 2}{6}}}$$

Find t: $A(t) = 4A_0$

$$A_0 e^{\frac{\ln 2}{6} t} = 4 A_0$$

$$e^{\frac{\ln 2}{6} t} = 4$$

$$\ln\left(e^{\frac{(\ln 2)t}{6}}\right) = \ln(4)$$

$$\frac{t(\ln 2)}{6} = \ln(4)$$

$$t = \frac{6 \ln(4)}{\ln 2} = \frac{6 \ln(2^2)}{\ln 2}$$

$$t = \frac{12 \ln(2)}{\ln(2)}$$

$$\boxed{t = 12}$$