03-06-25 MTH 263

3 Topics in Differential Calculus

3.1 Tangent Lines

page 71: 1, 5, 11, 15, 23, 27

3.2 Limits: Formal Definition

page 82: 1, 3, 5, 7, 13

Exam 2: 1.6, 2.1 -2.4, 3.1-3.2 omit big O

Thursday, 03/13/25

3.2: 7

For Exercises 1-18 evaluate the given limit.

7. $\lim_{x\to-\infty} x^2e^x$ $= \lim_{x\to-\infty} \frac{e^x}{\frac{1}{x^2}}$ $\overset{L}{=} \lim_{x\to-\infty} \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x^{-2})}$

$[\infty \cdot 0]$ $\left[\frac{0}{0}\right]$

$= \lim_{x\to-\infty} \frac{e^x}{-2x^{-3}} = -\frac{1}{2}\lim_{x\to-\infty} \frac{e^x}{\frac{1}{x^3}}$

$= -\frac{1}{2}\lim_{x\to-\infty} x^3e^x$ bad choice

7. $\lim_{x\to-\infty} x^2e^x$ $= \lim_{x\to-\infty} \frac{x^2}{e^{-x}}$ $\overset{L}{=}$

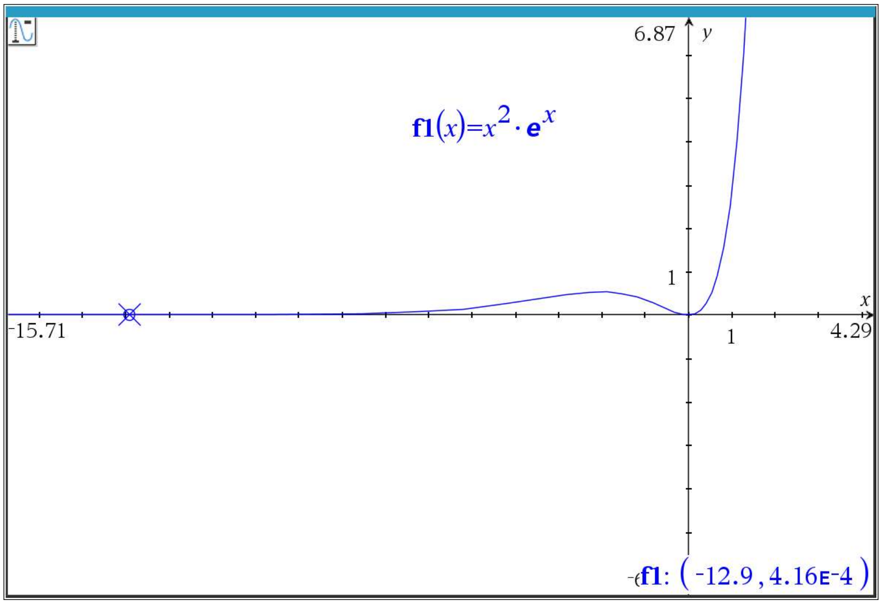
$\left[\frac{\infty}{\infty}\right]$

$\lim_{x\to-\infty} \frac{\frac{d}{dx}(x^2)}{\frac{d}{dx}(e^{-x})} = \lim_{x\to-\infty} \frac{2x}{-e^{-x}}$

$\left[\frac{-\infty}{-\infty}\right]$

$\overset{L}{=} \lim_{x\to-\infty} \frac{\frac{d}{dx}(2x)}{\frac{d}{dx}(-e^{-x})} = \lim_{x\to-\infty} \frac{2}{e^{-x}} = \boxed{0}$

$\frac{d}{dx}(-e^{-x}) \quad x \to -\infty \quad e$



Our graph suggests strongly that the limit is 0.

13. $\lim\limits_{x \to 0} \dfrac{\ln(1-x) - \sin^2 x}{1 - \cos^2 x}$

$$= \lim_{x \to 0} \frac{\ln(1-x) - \sin^2 x}{\sin^2 x}$$

$$= \lim_{x \to 0} \frac{\ln(1-x)}{\sin^2 x} - \lim_{x \to 0} 1$$

$$= \boxed{\lim_{x \to 0} \frac{\ln(1-x)}{\sin^2 x} - 1}$$

$$\lim_{x \to 0} \frac{\ln(1-x)}{\sin^2 x} \left[\frac{0}{0}\right] \overset{L}{=} \lim_{x \to 0} \frac{\frac{d}{dx}(\ln(1-x))}{\frac{d}{dx}(\sin^2 x)}$$

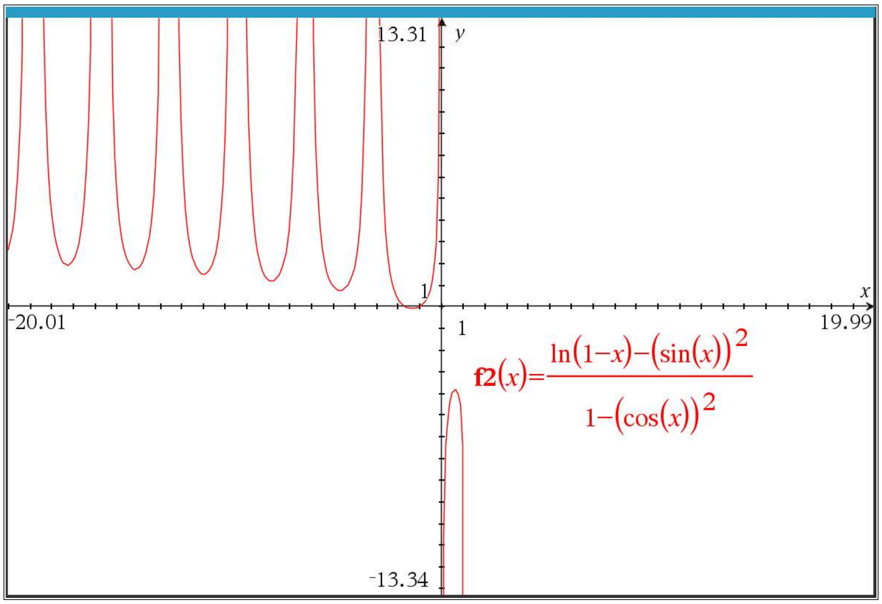
$[(\sin x)^2$

$$= \lim_{x \to 0} \frac{\frac{-1}{1-x}}{2(\sin x)(\cos x)}$$

$\lim \quad -1 \quad = \pm 1 \quad dne$

$$= \lim_{x \to 0} \frac{\overset{2|\sin x||\cos x|}{-1}}{(1-x)\sin^2 x} = \frac{\pm 1}{0} \quad \text{dne}$$

$$\Rightarrow \frac{\pm 1}{0} - 1 \quad \text{dne}$$



Find $\lim_{x \to 2} x^2$ and verify with ε-δ definition

$$\lim_{x \to 2} x^2 = 2^2 = \boxed{4}$$

Let $\varepsilon > 0$

Find $\delta > 0$ such that

$$0 < |x-2| < \delta \Rightarrow |x^2 - 4| < \varepsilon$$

$$|x^2 - 4| < \varepsilon$$

$$\Leftrightarrow |x+2||x-2| < \varepsilon$$

Strategy: find a constraint on $|x+2|$

Strategy: find a constraint on $|x+2|$

Let $|x-2|<1$ (arbitrary constant)

$\Rightarrow \quad -1 < x-2 < 1$

$4-1 < x-2+4 < 1+4$

$3 < x+2 < 5$

$\Rightarrow \quad |x+2| < 5 \quad \Leftrightarrow \quad -5 < x+2 < 5$

$|x+2||x-2| < 5|x-2|$

we would be done if $5|x-2| < \varepsilon$

$\Leftrightarrow \quad |x-2| < \frac{\varepsilon}{5}$

$$\boxed{\text{Let } \delta = \min\left(1, \frac{\varepsilon}{5}\right)}$$

e.g. if $\varepsilon = 10$, then $\delta = \min\left(1, \frac{10}{5}\right)$

$= \min(1, 2) = 1$

if $\varepsilon = .5$

Then $\delta = \min\left(1, \frac{.5}{5}\right) = \min(1, .1) = .1$

I claim $\lim_{x \to 1} (2x) = 6$

verify with ε-δ def

Let $\varepsilon > 0$

Find $\delta > 0$ such that

$0 < |x-1| < \delta \Rightarrow |2x-6| < \varepsilon$

$|2x-6| < \varepsilon \Leftrightarrow 2|x-3| < \varepsilon$

$\Leftrightarrow |x-3| < \frac{\varepsilon}{2}$

Let $\varepsilon = 2$

Find δ such that

$|x-1| < \delta \Rightarrow |x-3| < 1$

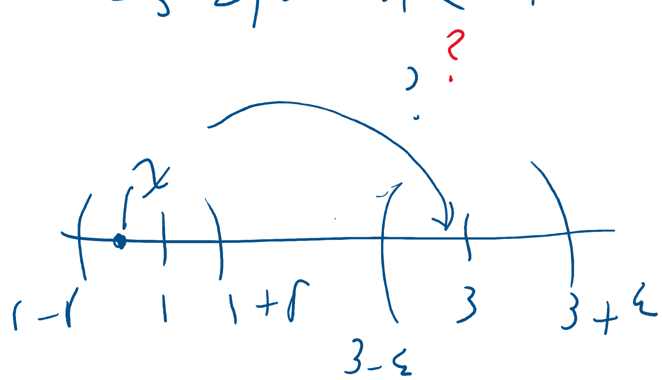
Try $\delta_1 = 1$

$|x-1| < 1$

$\Leftrightarrow -1 < x-1 < 1$

$\Leftrightarrow -1-2 < x-1-2 < 1-2$

$-3 < |x-3| < -1$



$\therefore \lim_{x \to 1} 2x \neq 6$

$\therefore \lim_{x \to 1} 2x \neq 6$

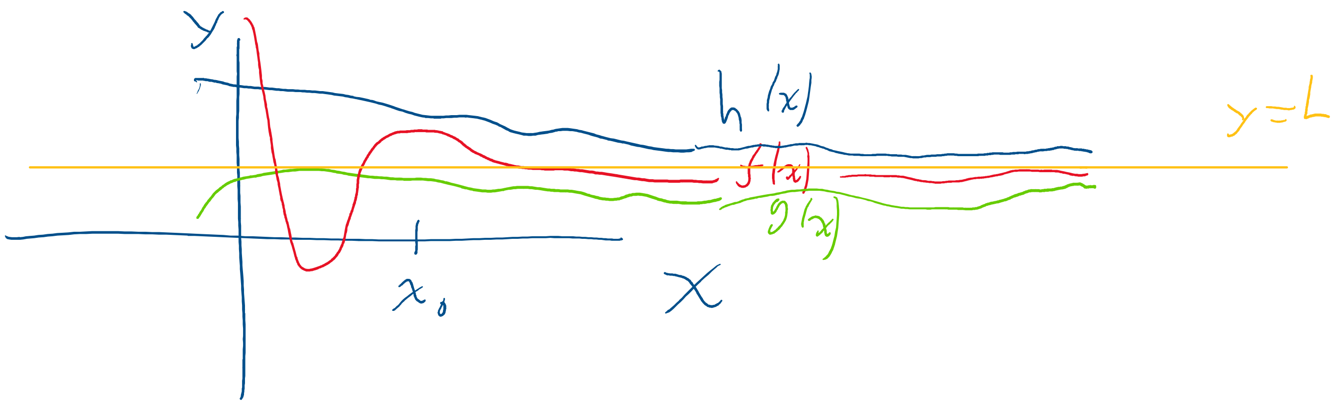
3.2

Squeeze Theorem: Suppose that for some functions f, g and h there is a number $x_0 \geq 0$ such that

$$g(x) \leq f(x) \leq h(x) \text{ for all } x > x_0$$

and that $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} h(x) = L$. Then $\lim_{x\to\infty} f(x) = L$.

Similarly, if $g(x) \leq f(x) \leq h(x)$ for all $x \neq a$ in some interval I containing a, and if $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} f(x) = L$.



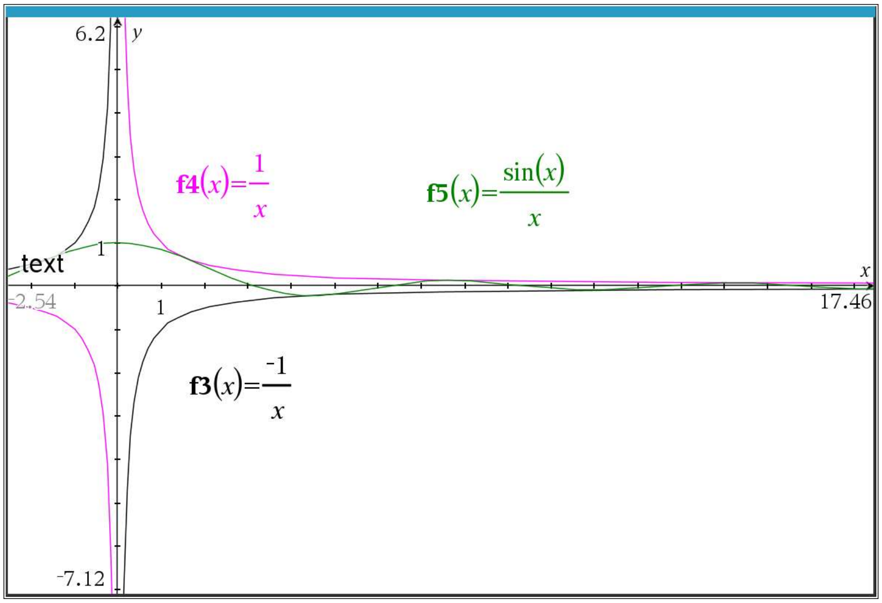
Example 3.21

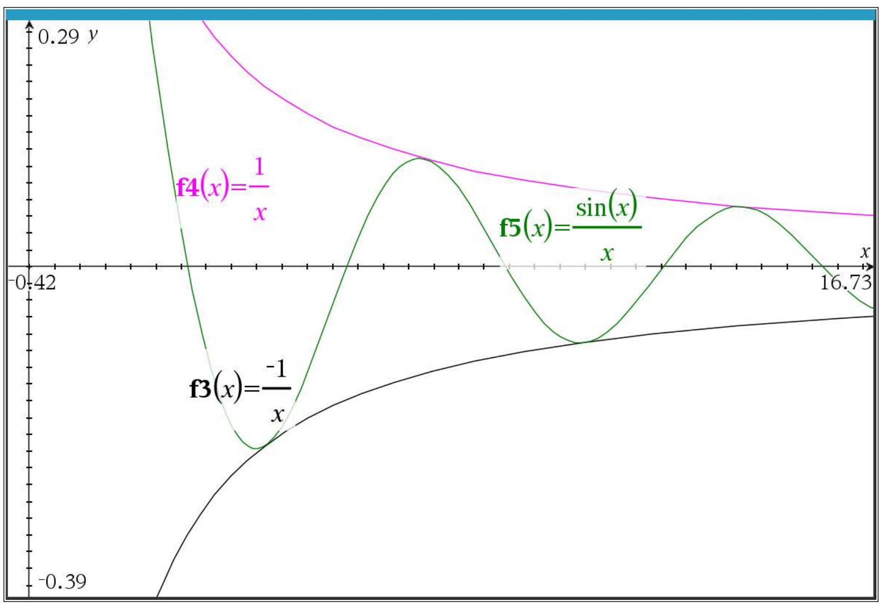
Evaluate $\lim_{x\to\infty} \dfrac{\sin x}{x}$.

Solution: Since $-1 \leq \sin x \leq 1$ for all x, then dividing all parts of those inequalities by $x > 0$ yields

$$-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x} \text{ for all } x > 0 \quad \Rightarrow \quad \lim_{x\to\infty} \frac{\sin x}{x} = 0$$

by the Squeeze Theorem, since $\lim_{x\to\infty} \dfrac{-1}{x} = 0 = \lim_{x\to\infty} \dfrac{1}{x}$.



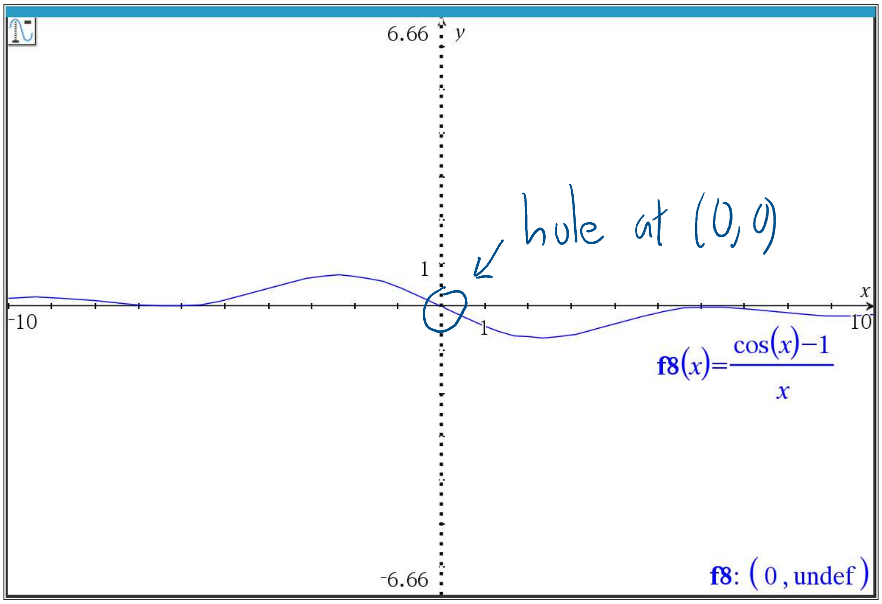


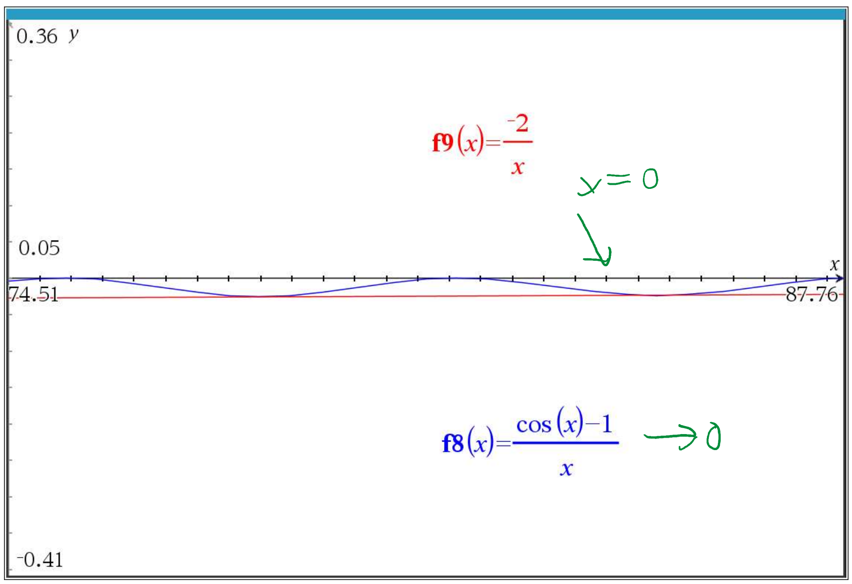
Guichard

$$\lim_{x\to 0}\frac{\cos x - 1}{x}. \quad \left[\frac{0}{0}\right] \overset{L}{=} \lim_{x\to 0}\frac{\frac{d}{dx}(\cos x - 1)}{\frac{d}{dx}(x)}$$

$$= \lim_{x\to 0}\frac{(-\sin x - 0)}{1}$$

$$= \lim_{x\to 0} -\sin x = -0 = \boxed{0}$$





squeeze theorem