03-04-25 MTH 263

2.4 General Exponential and Logarithmic Functions

page 65: 1, 3, 5, 7

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3 Topics in Differential Calculus

3.1 Tangent Lines

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3.2 Limits: Formal Definition

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Exam 2: 1.6, 2.1 -2.4, 3.1-3.2

Thursday, 03/13/25

3.1

Memorize

For a curve $y = f(x)$ that is differentiable at $x = a$, the **tangent line** to the curve at the point $P = (a, f(a))$ is the unique line through P with slope $m = f'(a)$. P is called the **point of tangency**. The equation of the tangent line is thus given by:

$$y - f(a) = f'(a)\cdot(x - a) \tag{3.1}$$

Supplied

The tangent line to a curve $y = f(x)$ makes an angle $\phi(x)$ with the positive x-axis, given by

$$\phi(x) = \tan^{-1} f'(x) . \tag{3.2}$$

$\tan(\varphi(x)) = f'(x)$

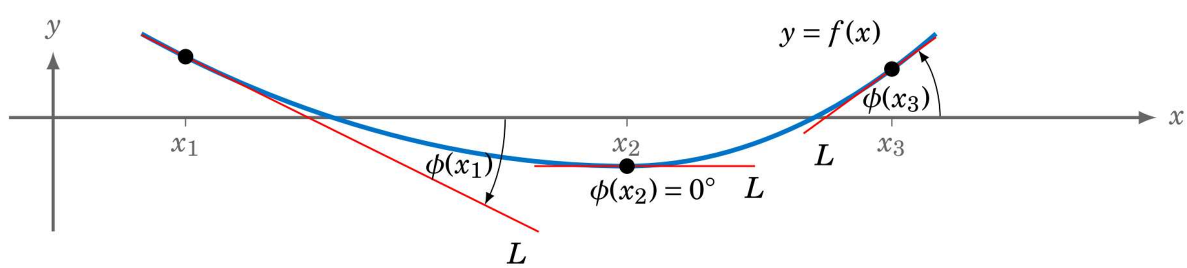
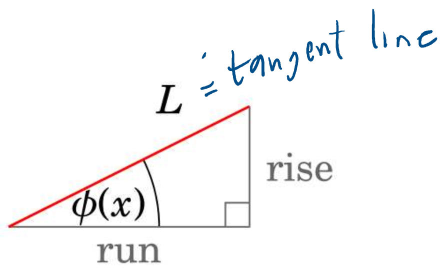


Figure 3.1.7 The angle $\phi(x)$ between the tangent line and positive x-axis



memorize

The equation of the normal line to a curve $y = f(x)$ at a point $P = (a, f(a))$ is

$$y - f(a) = -\frac{1}{f'(a)} \cdot (x - a) \quad \text{if } f'(a) \neq 0. \tag{3.3}$$

If $f'(a) = 0$, then the normal line is vertical and is given by $x = a$.

3.1

For Exercises 1-12, find the equation of the tangent line to the curve $y = f(x)$ at $x = a$.

$y - f(a) = f'(a) \cdot (x - a)$

2. $f(x) = x^2 - 1$; at $x = 2$

$y - f(2) = f'(2) \cdot (x - 2)$

$f(2) = 2^2 - 1 = 4 - 1 = 3$

Point of tangency $= (2, 3)$

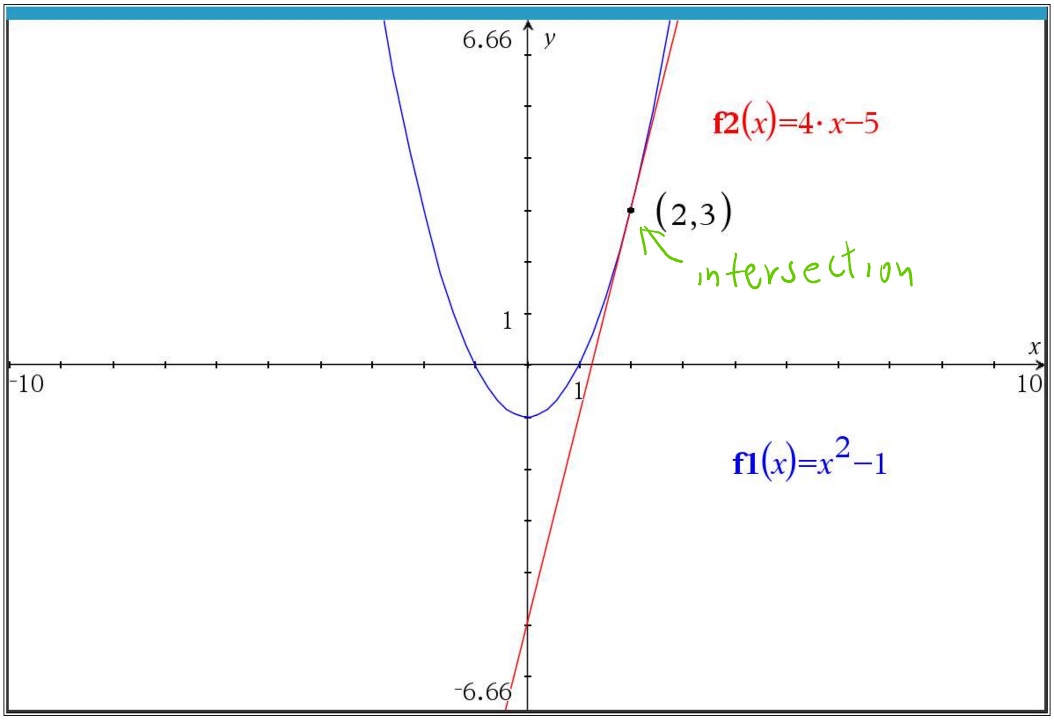
$f'(x) = 2x$

$\Rightarrow f'(2) = (2)(2) = 4$

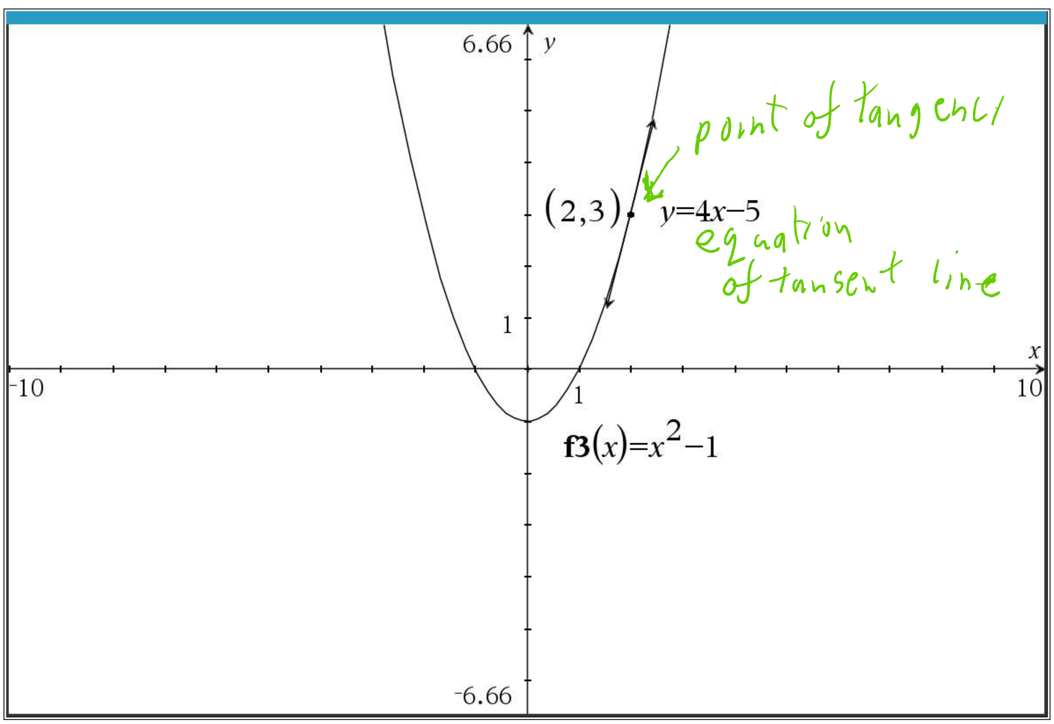
$\boxed{y - 3 = 4(x - 2)}$

$y = 4x - 8 + 3$

$\boxed{y = 4x - 5}$



TI-84 Graphed y = f(x), 2nd Draw - tan, 2, enter



3.2
Memorize

So far only the intuitive notion of a limit has been used, namely:

A real number L is the limit of $f(x)$ as x approaches a if the values of $f(x)$ can be made *arbitrarily* close to L by picking values of x *sufficiently* close to a.

memorize

Let L and a be real numbers. Then L is the **limit** of a function $f(x)$ as x approaches a, written as

$$\lim_{x \to a} f(x) = L,$$

if for any given number $\epsilon > 0$, there exists a number $\delta > 0$, such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

$\forall \varepsilon > 0, \exists \delta > 0$ such that

$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

$\forall$ = for any

$\exists$ = there exists

$\Rightarrow$ = implies

$\Leftrightarrow$ = is equivalent to

Find $\lim_{x \to 2} 3x - 5$

Verify limit by the formal ε-δ definition of limit

$$\lim_{x \to 2} (3x - 5) = 3(2) - 5 = 6 - 5 = \boxed{1}$$

Let $\varepsilon > 0$

Find $\delta > 0$ such that

$$0 < |x - 2| < \delta \Rightarrow |3x - 5 - 1| < \varepsilon$$

Strategy: work backwards

$$|3x - 5 - 1| < \varepsilon \Leftrightarrow |3x - 6| < \varepsilon$$

$$\Leftrightarrow 3|x - 2| < \varepsilon$$

$$\Leftrightarrow |x - 2| < \frac{\varepsilon}{3}$$

$$\Longleftrightarrow \quad |x-2| < \frac{\varepsilon}{3}$$

Let $\boxed{\delta = \frac{\varepsilon}{3}}$ $\quad \therefore \lim_{x \to 2} (3x-5) = 1$

no need to write this

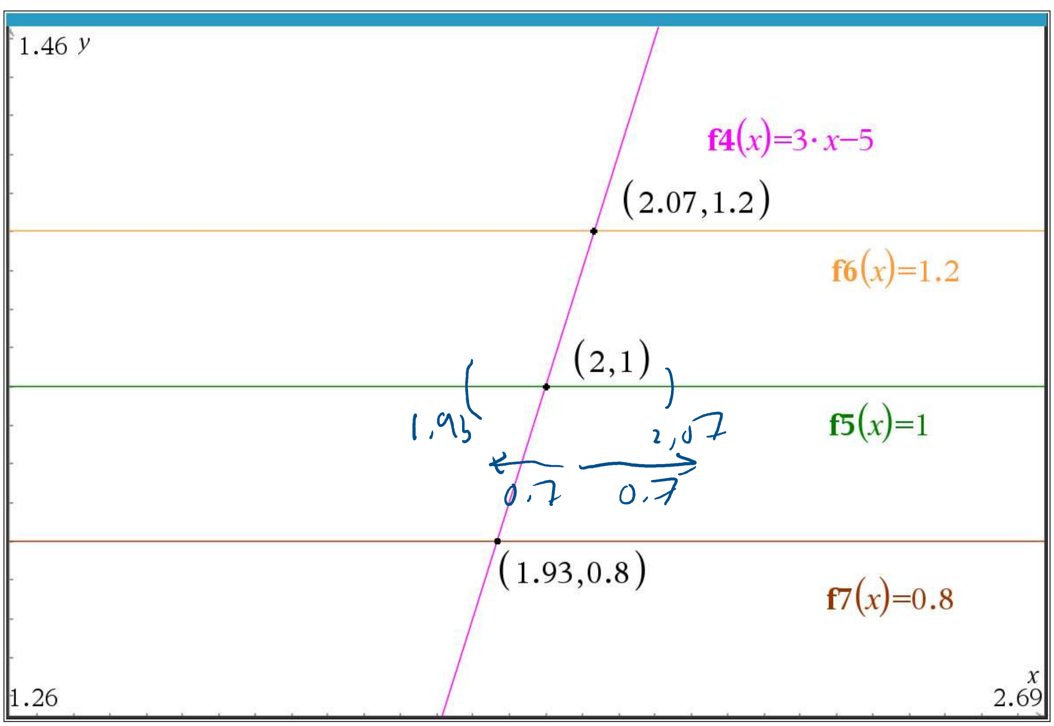
$$\left\{ \begin{aligned} &\delta = \frac{\varepsilon}{3} \Rightarrow |x-2| < \delta \Rightarrow |x-2| < \frac{\varepsilon}{3} \\ &\Rightarrow 3|x-2| < \varepsilon \\ &\Rightarrow |3x-6| < \varepsilon \\ &\Rightarrow |3x-5-1| < \varepsilon \end{aligned} \right.$$

Let $\varepsilon = 0.2$

Find δ on our calculator

From our calculation $\delta \approx \frac{0.2}{3} \approx 0.0667$

0.2/3=0.0667 ≈ 0.07



$2 - 1.93 = 0.07$

$2-1.93=0.07$

$2.07-2=0.07$

Let $\delta=0.07$

Memorize

Call L the **right limit** of a function $f(x)$ as x approaches a, written as

$$\lim_{x \to a+} f(x) = L\,,$$

if $f(x)$ approaches L as x approaches a for values of x larger than a.
Call L the **left limit** of a function $f(x)$ as x approaches a, written as

$$\lim_{x \to a-} f(x) = L\,,$$

if $f(x)$ approaches L as x approaches a for values of x smaller than a.

The limit of a function exists if and only if both its right limit and left limit exist and are equal:

$$\lim_{x \to a} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to a-} f(x) = L = \lim_{x \to a+} f(x)$$

TI-nspire

$$\lim_{x \to 0^+}\left(\frac{1}{x}\right) \qquad \infty$$

Memorize

L'Hôpital's Rule: If f and g are differentiable functions and

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty} \text{ or } \frac{0}{0}$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}\,.$$

The number a can be real, ∞, or $-\infty$.

$$\lim_{x \to \infty} \frac{x}{x} = \lim_{x \to \infty} 1 = 1$$

$$\left[\frac{\infty}{\infty}\right]$$

$$\lim_{x \to \infty} \frac{x^2}{x} = \lim_{x \to \infty} x = \infty$$

$$\left[\frac{\infty}{\infty}\right]$$

$$\lim_{x \to \infty} \frac{x}{x^2} = \lim_{x \to \infty} \frac{1}{x} = 0$$

$$\left[\frac{\infty}{\infty}\right]$$

function $\frac{f(x)}{g(x)}$ when $f(x) \to \pm\infty$, $g(x) \to \pm\infty$

or $f(x) \to 0$, $g(x) \to 0$ are <u>indeterminate</u>

$$\lim_{x \to \infty} \frac{x}{x} \underset{L}{=} \lim_{x \to \infty} \frac{\frac{d}{dx}(x)}{\frac{d}{dx}(x)}$$

$$\left[\frac{\infty}{\infty}\right]$$

apply L'Hopial's Rule

$$= \lim_{x \to \infty} \frac{1}{1} = \lim_{x \to \infty} 1 = \boxed{1}$$

$$\lim_{x\to\infty} 1 \qquad \lim_{x\to\infty}$$

$$\lim_{x\to\infty} \frac{x}{x^2} \underset{L}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}(x)}{\frac{d}{dx}(x^2)}$$

$$\left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to\infty} \frac{1}{2x} = 0$$

Example 3.19

Evaluate $\lim\limits_{x\to\infty} \dfrac{2x^2 - 7x - 5}{3x^2 + 2x - 1}$.

Solution: This limit is of the form ∞/∞:

$$\lim_{x\to\infty} \frac{2x^2 - 7x - 5}{3x^2 + 2x - 1} \to \frac{\infty}{\infty}$$

$$\left[\frac{\infty}{\infty}\right]$$

$$\underset{L}{=} \lim_{x\to\infty} \frac{4x - 7}{6x + 2} \quad \text{by L'Hôpital's Rule}$$

$$\to \frac{\infty}{\infty} \quad \text{, so use L'Hôpital's Rule again}$$

$$\underset{L}{=} \frac{4}{6} = \frac{2}{3}$$

$$\lim_{x\to\infty} \frac{2x^2 - 7x - 5}{3x^2 + 2x - 1}$$

$$= \lim_{x\to\infty} \frac{\frac{2x^2}{x^2} - \frac{7x}{x^2} - \frac{5}{x^2}}{\frac{3x^2}{x^2} + \frac{2x}{x^2} - \frac{1}{x^2}}$$

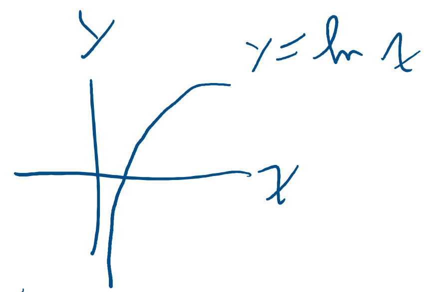
$$= \lim_{x\to\infty} \frac{2 - \frac{7}{x} - \frac{5}{x^2}}{3 + \frac{2}{x} - \frac{1}{x^2}}$$

$$= \frac{2 - 0 - 0}{3 + 0 - 0} = \boxed{\frac{2}{3}}$$

" 0^0 "

$$\lim_{x \to 0} x^x$$

$$\ln(x^x) = x \ln x$$



$$\lim_{x \to 0^+} x \ln x$$

$$[0 \cdot -\infty]$$

$$= \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} \overset{L}{=} \lim_{x \to 0^+} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x^{-1})}$$

$$\left[\frac{-\infty}{\infty}\right]$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-x^{-2}} = \lim_{x \to 0^+} \left(\frac{1}{x}\right)(-x^2)$$

$$= \lim_{x \to 0^+} (-x) = 0$$

$$\lim_{x \to 0^+} \ln(x^x) = \ln \lim_{x \to 0^+} (x^x) = 0$$

$$\Rightarrow \lim_{+} (x^x) = 1$$

$$\Rightarrow \lim_{x \to 0^+} \left(x^x\right) = 1$$

$$\lim_{x \to 0^+} \left(x^x\right) \qquad 1$$