02-25-25 MTH 263

2.2 Trigonometric Functions and Their Inverses

page 54: 1, 4, 5, 15

2.3 The Exponential and Natural Logarithm Functions

page 62: 1, 5, 7, 9, 13, 17, 19

2.2: 5

For Exercises 1-16, find the derivative of the given function $y = f(x)$.

5. $y = \tan^{-1}(x/3)$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{dy}{dx} = \frac{1}{\left(1+\left(\frac{x}{3}\right)^2\right)} \frac{d}{dx}\left(\frac{x}{3}\right)$$

$$= \frac{1}{1+\frac{x^2}{9}}\left(\frac{1}{3}\right)$$

$$= \frac{1}{3+\frac{x^2}{3}}$$

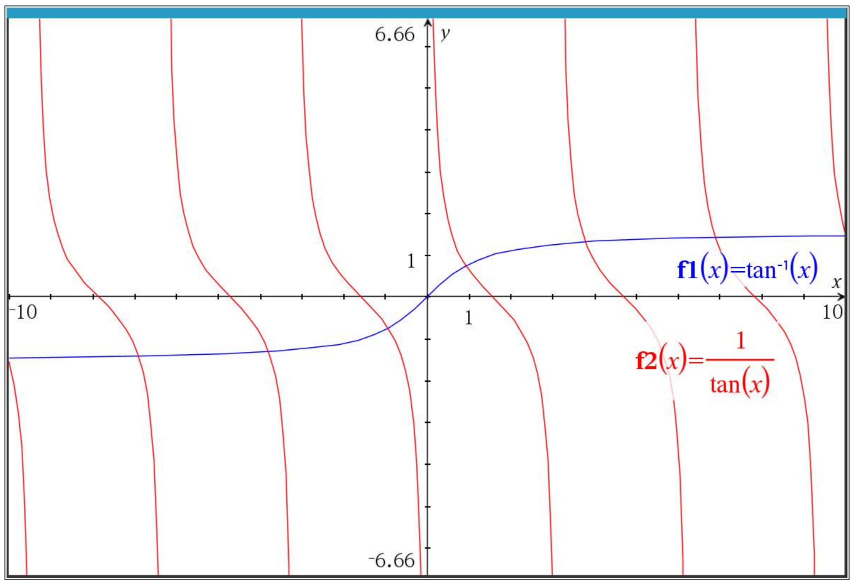
$$= \left(\frac{1}{3+\frac{x^2}{3}}\right)\left(\frac{3}{3}\right)$$

$$\boxed{\frac{dy}{dx} = \frac{3}{9+x^2}}$$

15. $y = x\cot^{-1}x$

$$\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

Note: $\tan^{-1}(x) \neq \frac{1}{\tan(x)}$



$$\frac{dy}{dx} = x\frac{d}{dx}\left(\cot^{-1}(x)\right) + \left(\frac{dx}{dx}\right)\cot^{-1}(x)$$

$$= x\left(-\frac{1}{1+x^2}\right) + (1)\cot^{-1}(x)$$

$$\boxed{\frac{dy}{dx} = -\frac{x}{1+x^2} + \cot^{-1}(x)}$$

Derive

$$\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

Let $y = \cot^{-1}(x)$

Find $\frac{dy}{dx}$

$$\cot y = \cot\left(\cot^{-1}(x)\right)$$

$$\cot y = x$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{d}{dy}(\cot y)}$$

$$\frac{d}{dy} \cot y = \frac{d}{dy}\left(\frac{\cos y}{\sin y}\right)$$

$$= \frac{\sin y \frac{d}{dy}(\cos y) - \cos y \frac{d}{dy}(\sin y)}{\sin^2 y}$$

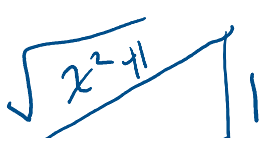
$$= \frac{\sin y(-\sin y) - \cos y \cos y}{\sin^2 y}$$

$$= \frac{-\sin^2 y - \cos^2 y}{\sin^2 y} = \frac{-(\sin^2 y + \cos^2 y)}{\sin^2 y}$$

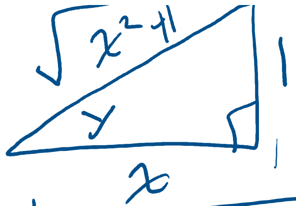
$$= -\frac{1}{\sin^2 y} = -\csc^2 y$$

$$\frac{dy}{dx} = \frac{1}{-\csc^2 y} = -\sin^2 y$$

$$x = \cot y = \frac{adj}{opp} = \frac{x}{1}$$



$$\Rightarrow \sin y = \frac{opp}{} = \frac{1}{}$$



$\Rightarrow \sin y = \frac{\text{opp}}{\text{hyp}} = \frac{1}{\sqrt{x^2+1}}$

$$\boxed{\frac{dy}{dx} = -\frac{1}{x^2+1}}$$

Implicit differentiation

$$x = \cot y$$

$$\frac{dx}{dx} = \frac{d}{dx}\left(\cot y(x)\right)$$

$$1 = \left(-\csc^2(y)\right)\frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{\csc^2(y)}$$

$$\frac{dy}{dx} = -\sin^2(y)$$

$$\frac{dy}{dx} = -\frac{1}{1+x^2}$$

1.2:23

23. Suppose that $f(a+b) = f(a)f(b)$ for all a and b, and $f'(0)$ exists. Show that $f'(x)$ exists for all x.

$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ if the limit exists

$= \lim_{h\to 0} \frac{f(x)f(h) - f(x)}{h}$ if the limit exists

$= \lim_{h\to 0} \frac{f(x)\left[f(h) - 1\right]}{}$ if the limit exists

$= \lim_{h \to 0} \frac{f(x)[f(h)-1]}{h}$ if the limit exists

$= f(x) \lim_{h \to 0} \left[\frac{f(h)-1}{h} \right]$ if the limit exists

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$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$ exists

$= \lim_{h \to 0} \frac{f(h) - f(0)}{h}$ exists

we would be done if $f(0) = 1$

case 1: assume $f(1) \neq 0$

$f(1+0) = f(1) f(0)$

$\Rightarrow f(1) = f(1) f(0)$

$\Rightarrow 1 = f(0)$

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case 2: $f(1) = 0$

$f(a+1) = f(a) f(1) = (f(a))(0) = 0$

for any $a \in \mathbb{R}$

$\Rightarrow f(x) = 0$ all $x \in \mathbb{R}$

$\therefore f$ is constant

$\Rightarrow f'(x) = 0$ all $x \Rightarrow f'(x)$ exists for all x

QED

Quiz 3

Let $y = \cos(\tan(x))$.

Find $\frac{dy}{dx}$.

Scientific Notebook

$\frac{d}{dx}(\cos(\tan x)) = -(\sin(\tan x))(\tan^2 x + 1)$

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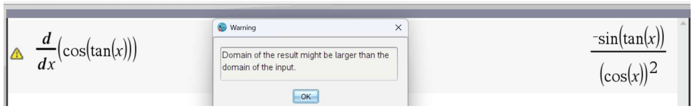

$$\frac{dy}{dx} = -\sin(\tan x)\,\frac{d}{dx}(\tan x)$$
$$= -\sin(\tan x)(\sec^2 x)$$

2.3

$$e = \lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x \qquad (2.1)$$

The approximate value of is $e = 2.71828182845905\ldots$ (often called the *Euler number*).

$$e \approx \left(1 + \frac{1}{x}\right)^x \quad \Rightarrow \quad e^{1/x} \approx \left(\left(1 + \frac{1}{x}\right)^x\right)^{1/x} = 1 + \frac{1}{x} \quad \Rightarrow \quad \left(e^{1/x} - 1\right)x \approx \left(\frac{1}{x}\right)x = 1\,,$$

Let $h = \frac{1}{x} \Rightarrow x = \frac{1}{h}$

$$(e^h - 1)\left(\frac{1}{h}\right) \approx 1$$

so letting $h = 1/x$, and noting that $h = 1/x \to 0$ if and only if $x \to \infty$, yields the useful limit:[5]

$$\lim_{h\to 0}\frac{e^h - 1}{h} = 1$$

Memorize

$$\frac{d}{dx}\left(e^x\right) = e^x$$

domain of $\ln x$ = all $x > 0$ = range of e^x

$$\begin{aligned} \text{domain of } \ln x \quad &= \quad \text{all } x > 0 \quad = \quad \text{range of } e^x \\ \text{range of } \ln x \quad &= \quad \text{all } x \quad\quad\;\; = \quad \text{domain of } e^x \end{aligned}$$

$$y = e^x \quad \text{if and only if} \quad x = \ln y$$

$$e^{\ln x} = x \quad \text{for all } x > 0$$

$$\ln\left(e^x\right) = x \quad \text{for all } x$$

Memorize

$$\ln(ab) = \ln a + \ln b \qquad\qquad e^a \cdot e^b = e^{a+b}$$

$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b \qquad\qquad \frac{e^a}{e^b} = e^{a-b}$$

$$\ln a^b = b \ln a \qquad\qquad \left(e^a\right)^b = e^{ab}$$

$$\ln 1 = 0 \qquad\qquad e^0 = 1$$

Memorize

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Memorize

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$$