02-20-25 MTH 263

1.6 Higher Order Derivatives
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2 Derivatives of Common Functions

2.1 Inverse Functions
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2.2 Trigonometric Functions and Their Inverses
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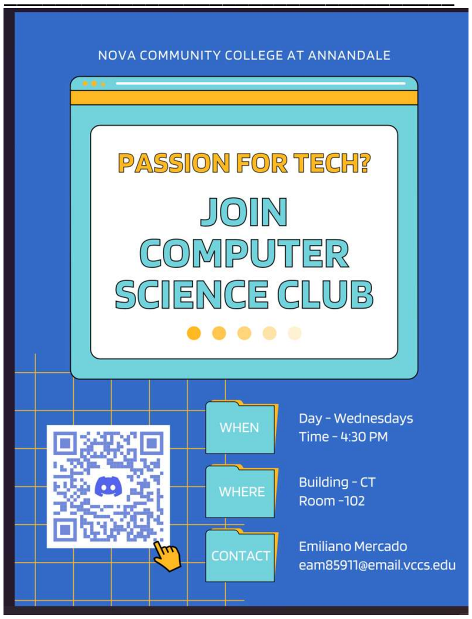
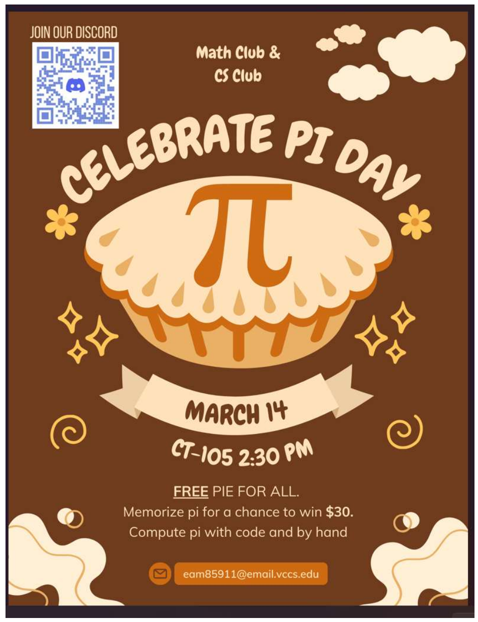
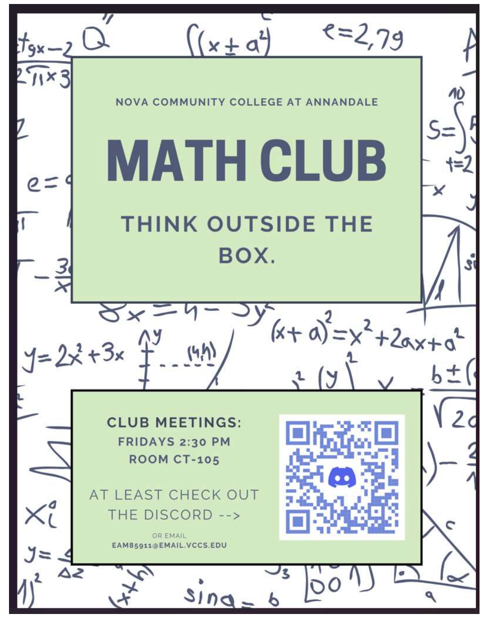
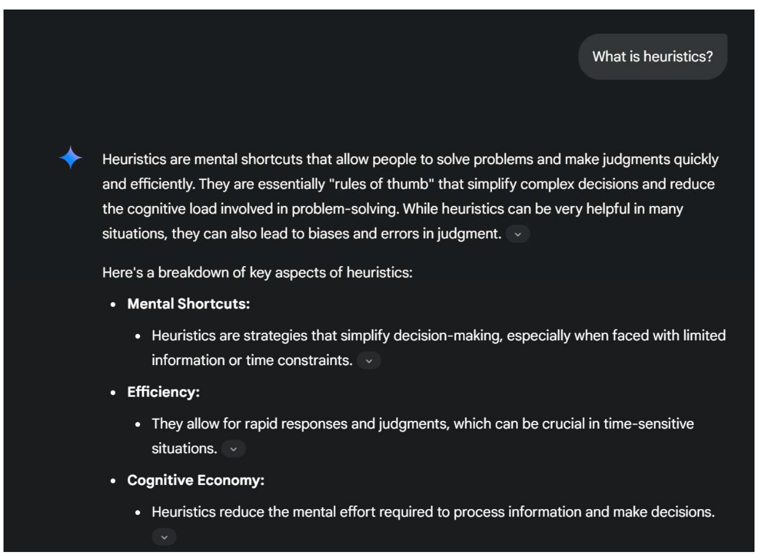








- **Potential for Bias:**
  - Because they are simplifications, heuristics can lead to systematic errors in judgment, known as cognitive biases.
- **Types of Heuristics:**
  - Several different types of heuristics have been identified, including:
    - **Availability heuristic:** Judging the likelihood of an event based on how easily examples come to mind.
    - **Representativeness heuristic:** Judging the likelihood of an event based on how similar it is to a typical case.
    - **Anchoring and adjustment heuristic:** Relying heavily on the first piece of information offered (the "anchor") when making decisions.

In essence, heuristics are a trade-off between speed and accuracy. They are valuable tools for navigating the complexities of everyday life, but it's important to be aware of their potential limitations.

heuristics and Polya

George Pólya, a renowned mathematician and educator, significantly contributed to our understanding of problem-solving. While he didn't explicitly use the term "heuristics" in the same way modern cognitive psychology does, his work is deeply rooted in the concept of heuristic problem-solving.

Here's how Pólya's work relates to heuristics:

- **Pólya's Problem-Solving Framework:**
  - Pólya's famous book, "How to Solve It," outlines a four-step process for problem-solving:
    1. **Understanding the problem:** Clearly defining what is known and what needs to be found.
    2. **Devising a plan:** Developing a strategy to solve the problem.
    3. **Carrying out the plan:** Implementing the chosen strategy.
    4. **Looking back:** Reviewing the solution and the process.
  - Within this framework, Pólya emphasizes the importance of using "heuristic reasoning" to guide the problem-solving process.
- **Heuristic Reasoning in Pólya's Work:**
  - Pólya's "heuristic reasoning" involves using general strategies and techniques to explore possible solutions. These strategies are essentially heuristics in the broader sense.
  - Examples of Pólya's heuristic strategies include:
    - **Looking for patterns:** Identifying recurring patterns in the problem.
    - **Drawing diagrams:** Visualizing the problem to gain a better understanding.
    - **Considering special cases:** Examining simpler versions of the problem.
    - **Working backward:** Starting with the desired outcome and working backward to the initial conditions.
    - **Finding a related problem:** Using the solution of a similar problem as a guide.
  - These are all heuristic techniques because they are not guaranteed to provide a solution, but they increase the likelihood of finding one.
- **Connection to Modern Heuristics:**
  - Pólya's work laid the foundation for later research on heuristics in cognitive psychology.
  - His emphasis on general problem-solving strategies aligns with the concept of heuristics as mental shortcuts.
  - While modern research has further explored specific types of heuristics and their associated biases, Pólya's work remains a valuable contribution to our understanding of how people solve problems.

In summary, Pólya's problem-solving framework is essentially a guide to using effective heuristics in mathematical problem-solving. His work highlights the importance of using general strategies and techniques to explore possible solutions, which is the core concept of heuristic reasoning.

What are some good textbooks that introduce methods of mathematical proof?

Several excellent textbooks introduce methods of mathematical proof, catering to different levels and approaches. Here's a breakdown of some highly regarded options:

For Introductory/Transition to Proof Courses:

- **"How to Prove It: A Structured Approach" by Daniel J. Velleman:**
  - This is a classic and widely used textbook.
  - It emphasizes a clear, step-by-step approach to constructing proofs.
  - Covers propositional logic, quantifiers, set theory, relations, functions, and various proof techniques (direct proof, contrapositive, contradiction, induction).
  - Known for its readability and abundance of examples and exercises.
- **"Book of Proof" by Richard Hammack:**
  - A free, open-source textbook available online.
  - Covers similar topics to Velleman's book but with a slightly different style.
  - Excellent for self-study and provides many exercises.
  - It is available as a pdf, and is very well written.

- **"Mathematical Thinking: Problem-Solving and Proofs" by John P. D'Angelo and Douglas B. West:**
  - Combines problem-solving strategies with proof techniques.
  - Focuses on developing mathematical intuition and creativity.
  - Suitable for students who want a broader perspective on mathematical thinking.
- **"A Transition to Advanced Mathematics" by Douglas Smith, Maurice Eggen, and Richard St. Andre:**
  - A comprehensive textbook covering a wide range of topics, including logic, set theory, relations, functions, and number theory.
  - Provides a solid foundation for more advanced mathematics courses.

For More Advanced/Theoretical Approaches:

- **"Naive Set Theory" by Paul Halmos:**
  - While not strictly about proof techniques, it provides a rigorous introduction to set theory, which is fundamental to many mathematical proofs.
  - Written in an elegant and accessible style.
- **"Logic and Proofs" by Roberto Cori and Daniel Lascar:**
  - This is a more theoretical approach, and is very good for those that want to understand the underpinnings of mathematical logic.

Key Features to Look For:

- **Clear explanations:** The book should explain concepts in a way that is easy to understand.
- **Plenty of examples:** Examples help illustrate the concepts and techniques.
- **Exercises with varying difficulty:** Exercises allow you to practice and solidify your understanding.
- **Emphasis on logical reasoning:** The book should stress the importance of logical precision and clarity.

When choosing a textbook, consider your current mathematical background and the level of rigor you're seeking. For most students transitioning to proof-based mathematics, Velleman's "How to Prove It" or Hammack's "Book of Proof" are excellent starting points.

2.1 supplied

$$\left(f^{-1}\right)'(f(x)) = \frac{1}{f'(x)} \qquad \text{if } f'(x) \neq 0$$

Two equivalent ways to write this are:

$$\left(f^{-1}\right)'(c) = \frac{1}{f'(a)} \qquad \text{where } c = f(a) \text{ and } f'(a) \neq 0$$

and

$$\left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))} \qquad \text{if } f'(f^{-1}(x)) \neq 0$$

$$\left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))} \qquad \text{if } f'(f^{-1}(x)) \neq 0$$

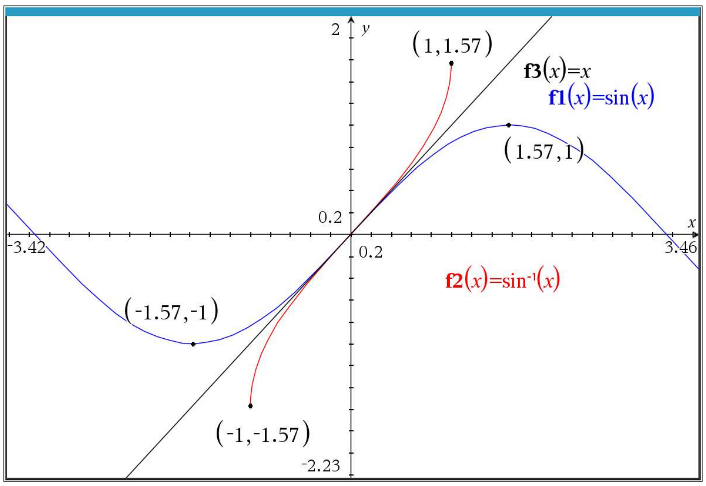
2.1: 10

10. Show that if f is differentiable and invertible, and if f^{-1} is twice-differentiable, then

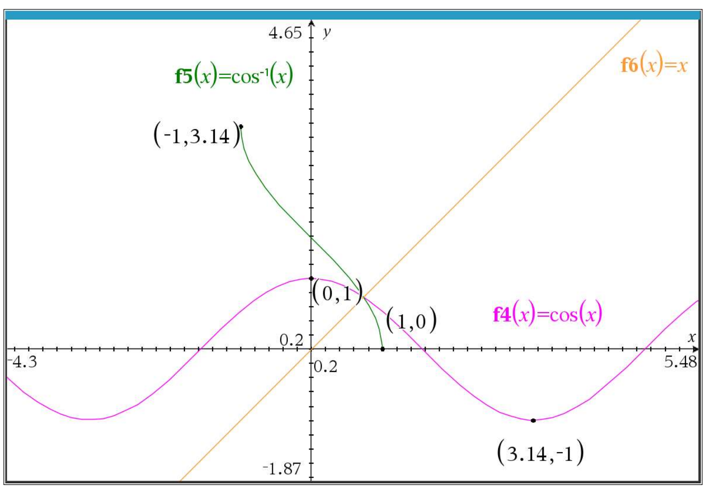
$$\left(f^{-1}\right)''(x) = -\frac{f''(f^{-1}(x))}{\left(f'(f^{-1}(x))\right)^3}.$$

Strategy: Clearly, the left side of the equation in this problem is the first derivative of the third formula in the above box. Therefore, we need to take the first derivative of the right side.

2.2



In order to define inverse trig functions,
We agree to restrict the domains of the trig functions to intervals on which they are 1-1.



memorize

$$\boxed{\begin{aligned}\frac{d}{dx}(\sin x) &= \cos x \\ \frac{d}{dx}(\cos x) &= -\sin x\end{aligned}} \qquad \begin{aligned}\frac{d}{dx}(\csc x) &= -\csc x \cot x \\ \frac{d}{dx}(\sec x) &= \sec x \tan x\end{aligned}$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Supplied

function	$\sin^{-1}x$	$\cos^{-1}x$	$\tan^{-1}x$	$\csc^{-1}x$	$\sec^{-1}x$	$\cot^{-1}x$
domain	$[-1,1]$	$[-1,1]$	$(-\infty,\infty)$	$\lvert x\rvert \geq 1$	$\lvert x\rvert \geq 1$	$(-\infty,\infty)$
range	$[-\frac{\pi}{2},\frac{\pi}{2}]$	$[0,\pi]$	$(-\frac{\pi}{2},\frac{\pi}{2})$	$(-\frac{\pi}{2},0)\cup(0,\frac{\pi}{2})$	$(0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi)$	$(0,\pi)$

The derivatives of the six inverse trigonometric functions are:

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \quad (\text{for } |x|<1) \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{|x|\sqrt{x^2-1}} \quad (\text{for } |x|>1)$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \quad (\text{for } |x|<1) \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}} \quad (\text{for } |x|>1)$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

Prove

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \quad (\text{for } |x|<1)$$

$y = \cos^{-1}(x)$

$\Leftrightarrow \boxed{\cos y = x}$

$\frac{d}{dy}(\cos y) = \frac{dx}{dy}$

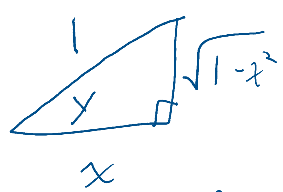
$\boxed{-\sin y = \frac{dx}{dy}}$

$\frac{d}{dx}(\cos^{-1}x) = \frac{dy}{dx}$

$= \frac{1}{\frac{dx}{dy}} = \frac{1}{-\sin y}$

$\sin y = \sqrt{1-\cos^2 y}$

$= \sqrt{1-x^2}$



$\cos y = x = \frac{x}{1} = \frac{adj}{hyp}$

$\sin y = \frac{opp}{hyp} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$

$\Rightarrow \boxed{\frac{d}{dx}(\cos^{-1}(x)) = -\frac{1}{\sqrt{1-x^2}}}$

Implicit differentiation

Prove

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \quad (\text{for } |x| < 1)$$

Let $y = \cos^{-1} x$ Goal: Find $\frac{dy}{dx}$

$\Rightarrow \cos y = x$

Think: $y = y(x)$

$$\frac{d}{dx}\left(\cos(y(x))\right) = \frac{dx}{dx}$$

$$-\sin(y(x))\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = -\frac{1}{\sin(y)}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\cos^2 y}}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$\sin^2 y + \cos^2 y = 1$

$\sin^2 y = 1 - \cos^2 y$

$\sin y = \sqrt{1-\cos^2 y}$

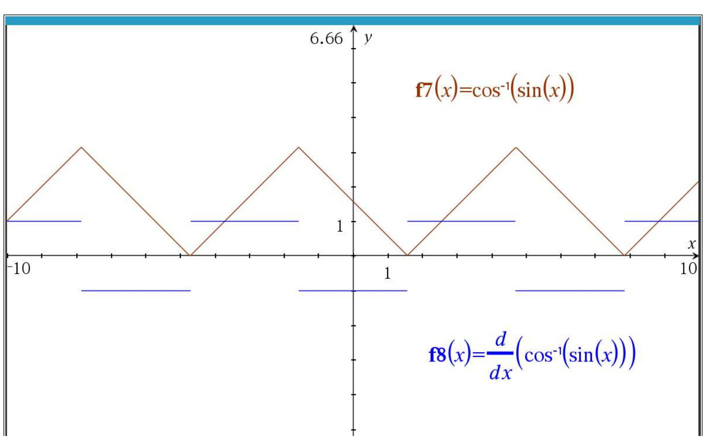
$$\boxed{\frac{d}{dx}\left(\cos^{-1}(x)\right) = -\frac{1}{\sqrt{1-x^2}}}$$

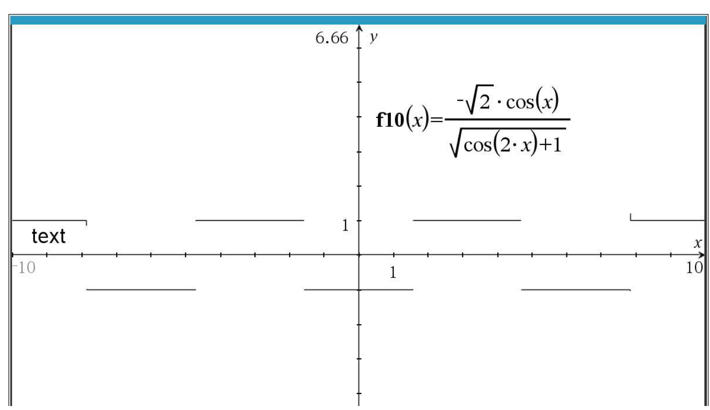
2.2

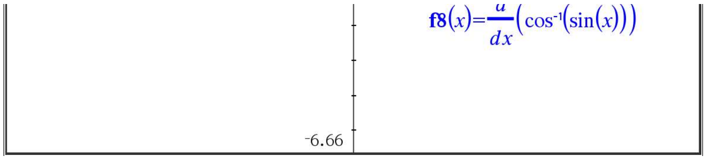
A

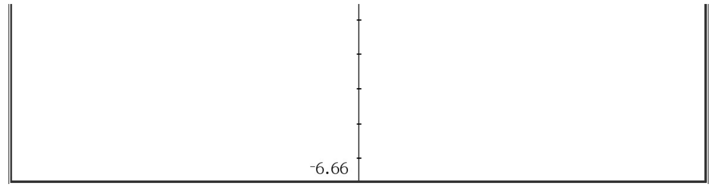
For Exercises 1-16, find the derivative of the given function $y = f(x)$.

8. $y = \cos^{-1}(\sin x)$









$$\frac{dy}{dx} = \frac{d}{dx}\left(\cos^{-1}(\sin x)\right)$$

$$= \left(\frac{-1}{\sqrt{1-\sin^2 x}}\right)\left(\frac{d}{dx}(\sin x)\right)$$

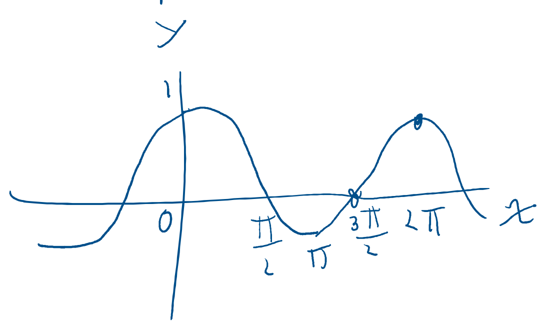
$$= \left(\frac{-1}{\sqrt{\cos^2 x}}\right)(\cos x)$$

$$= \left(\frac{-1}{|\cos x|}\right)(\cos x)$$

$$\boxed{\frac{dy}{dx} = \frac{-\cos x}{|\cos x|}}$$

$\sqrt{x^2} = |x|$

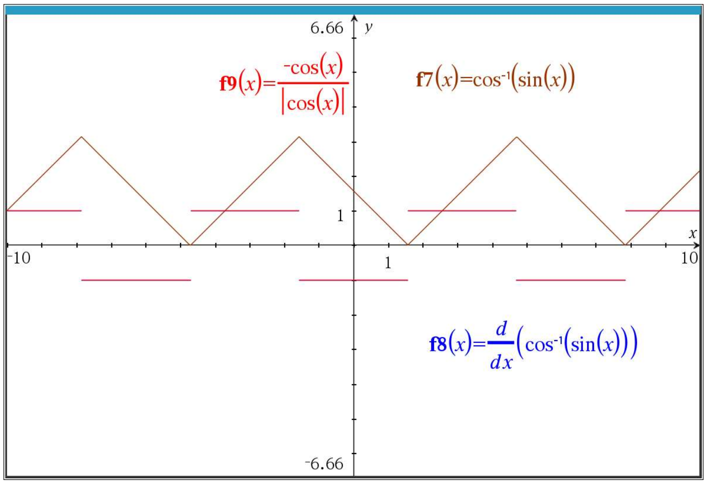
$\sqrt{(-3)^2} = \sqrt{9}$
$= 3$



$0 \le x \le \frac{\pi}{2}$

$\Rightarrow \frac{dy}{dx} = \frac{-\cos x}{\cos x} = -1$

$\frac{\pi}{2} \le x \le \frac{3\pi}{2} \Rightarrow \frac{dy}{dx} = -\frac{\cos x}{-\cos x} = 1$



Strategy for dealing with inverse functions: give a name to the inverse function

$y = \cos(\sin^{-1}(x))$

Let $w = \sin^{-1}(x)$

$\Rightarrow y = \cos w$

$y = \sin^{-1}(\cos x)$

$\sin y = \sin(\sin^{-1}(x))$

$\sin y = \cos x$

$\sqrt{4} = 2$

solve $x^2 = 4$

$x = \pm\sqrt{4}$

$x = \pm 2$

Google Gemini

What is the derivative of arccos(sin(x))?

Let's find the derivative of arccos(sin(x)).

1. Apply the Chain Rule

Let y = arccos(sin(x)). We need to find dy/dx.

We'll use the chain rule: dy/dx = dy/du * du/dx, where u = sin(x).

- **dy/du:** The derivative of arccos(u) is -1 / sqrt(1 - u^2).
- **du/dx:** The derivative of sin(x) is cos(x).

Therefore,

dy/dx = (-1 / sqrt(1 - (sin(x))^2)) * cos(x)

2. Simplify the Expression

We know that 1 - sin^2(x) = cos^2(x). So,

dy/dx = (-1 / sqrt(cos^2(x))) * cos(x)

dy/dx = (-1 / |cos(x)|) * cos(x)

3. Consider the Absolute Value

- If cos(x) > 0, then |cos(x)| = cos(x), and dy/dx = (-1 / cos(x)) * cos(x) = -1.
- If cos(x) < 0, then |cos(x)| = -cos(x), and dy/dx = (-1 / -cos(x)) * cos(x) = 1.

Therefore,

dy/dx = -cos(x)/|cos(x)|

Which can also be written piecewise as:

- dy/dx = 1, when cos(x) < 0
- dy/dx = -1, when cos(x) > 0

4. Alternative approach

We can also use the fact that sin(x) = cos(π/2 - x).

Then arccos(sin(x)) = arccos(cos(π/2 - x)).

Now, arccos(cos(θ)) = θ, provided that 0 ≤ θ ≤ π.

So, arccos(cos(π/2 - x)) = π/2 - x, when 0 ≤ π/2 - x ≤ π.

Which is equivalent to -π/2 ≤ -x ≤ π/2, or -π/2 ≤ x ≤ π/2.

Therefore, d/dx (π/2 - x) = -1.

However, the result of arccos(sin(x)) is a sawtooth wave. So the derivative will be 1 or -1 depending upon the quadrant of the x value.

In summary, the derivative of arccos(sin(x)) is:

- -1 when cos(x) > 0
- 1 when cos(x) < 0

Which can be written as:

dy/dx = -cos(x)/|cos(x)|.

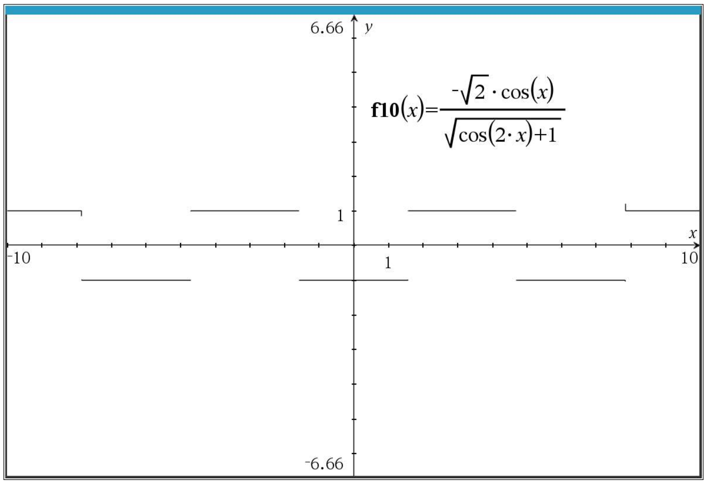
Scientific Notebook

$$\frac{d}{dx}(\arccos(\sin(x)) = -\sqrt{2}\,\frac{\cos x}{\sqrt{\cos 2x+1}}$$

When I tried Check equality, SN said

$$-\sqrt{2}\,\frac{\cos x}{\sqrt{\cos 2x+1}} = \frac{-\cos(x)}{|\cos(x)|} \text{ is false}$$

However, when I graphed SN' s answer,
I obtained the correct graph.



6.66
y
$\mathbf{f10}(x)=\frac{-\sqrt{2}\cdot\cos(x)}{\sqrt{\cos(2\cdot x)+1}}$
1
x
-10
1
10
-6.66