02-18-25 MTH 263

1.6 Higher Order Derivatives

page 45: 1, 3, 7, 9

2 Derivatives of Common Functions

2.1 Inverse Functions

page 50: 1, 3, 5

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Exam 1		stem & leaf		
53.16667	mean			A-0
47	median	8	58	B-2
20.73979	st. dev	7	8	C-1
20	min	6	3	D-1
88	max	5	09	F- 8
12	count	4	24	
		3	379	
		2	0	

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1.6

Memorize

Notation for the second derivative of $y = f(x)$: The following are all equivalent:

$$f''(x)\ ,\quad f^{(2)}(x)\ ,\quad \frac{d^2y}{dx^2}\ ,\quad \frac{d^2}{dx^2}(f(x))\ ,\quad y''\ ,\quad y^{(2)}\ ,\quad \ddot{y}\ ,\quad \ddot{f}(x)\ ,\quad \frac{d^2f}{dx^2}\ ,\quad D^2f(x)$$

Notation for the n-th derivative of $y = f(x)$: The following are all equivalent:

$$f^{(n)}(x)\ ,\quad \frac{d^ny}{dx^n}\ ,\quad \frac{d^n}{dx^n}(f(x))\ ,\quad y^{(n)}\ ,\quad \frac{d^nf}{dx^n}\ ,\quad D^nf(x)$$

memorize

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^2}{dx^2}\left(\frac{dy}{dx}\right)$$

$$\vdots$$

$$\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^{n-1}}{dx^{n-1}}\left(\frac{dy}{dx}\right)$$

Memorize

$$s(t) = \text{position at time } t$$

$$v(t) = \text{velocity at time } t$$

$$= \frac{ds}{dt} = s'(t) = \dot{s}(t)$$

$$a(t) = \text{acceleration at time } t$$

$$= \frac{dv}{dt} = v'(t) = \dot{v}(t)$$

$$= \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2} = s''(t) = \ddot{s}(t)$$

Supplied

$$\frac{d^n}{dx^n}(x^n) = n! \quad \text{for all integers } n \geq 0$$

Supplied

$$\frac{d^{n+1}}{dx^{n+1}}\left(x^n\right) = \frac{d}{dx}\left(\frac{d^n}{dx^n}\left(x^n\right)\right) = \frac{d}{dx}(n!) = 0$$

The $(n+1)$-st derivative ("n plus first derivative") of a polynomial of degree n is 0:
For any polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ of degree n, $\frac{d^{n+1}}{dx^{n+1}}(p(x)) = 0$.

Prove by math induction

$$\frac{d^n}{dx^n}(x^n) = n! \quad \text{for all integers } n \geq 0$$

basis step let $n=0$, Assume $x \neq 0$

$$\frac{d^0}{dx^0}(x^0) \stackrel{?}{=} 0!$$

$$x^0 \stackrel{?}{=} 1$$

$$1 = 1$$

Inductive hypothesis

Assume $\frac{d^n}{dx^n}(x^n) = n!$ for any fixed $n \geq 0, n \in \mathbb{Z}$

Prove $\frac{d^{n+1}}{dx^{n+1}}(x^{n+1}) \stackrel{?}{=} (n+1)!$

$$\frac{d^n}{dx^n}(x^n) = n!$$

$$\frac{d}{dx}\left(\frac{d^n}{dx^n}(x^n)\right) = \frac{d}{dx}(n!) \quad \text{def of higher derivative}$$

$$\frac{d}{dx}(n!) = \frac{d}{dx}(n!) \quad \text{Ind hyp}$$

True

QED.

2.1

Definition: a function f(x) is one-to-one (1-1) if
$f(c) = f(d) \Rightarrow c = d$

Derivative of an Inverse Function: If $y = f(x)$ is differentiable and has an inverse function $x = f^{-1}(y)$, then f^{-1} is differentiable and its derivative is

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \qquad \text{if } \frac{dy}{dx} \neq 0.$$

Example 2.1

Find the inverse f^{-1} of the function $f(x) = x^3$ then find the derivative of f^{-1}.

Solution: The function $y = f(x) = x^3$ is one-to-one over the set of all real numbers (why?) so it has an inverse function $x = f^{-1}(y)$ defined for all real numbers, namely $x = f^{-1}(y) = \sqrt[3]{y}$.

The derivative of f^{-1} is

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{3x^2}$$, which is in terms of x, so putting it in terms of y yields

$$= \frac{1}{3\left(\sqrt[3]{y}\right)^2} = \frac{1}{3y^{2/3}}$$

$x = y^{1/3}$

$\frac{dx}{dy} = \left(\frac{1}{3}\right)\left(y^{-2/3}\right) = \frac{1}{3y^{2/3}}$

which agrees with the derivative obtained by differentiating $x = \sqrt[3]{y}$ directly. Note that this derivative is defined for all y except $y = 0$, which occurs when $x = \sqrt[3]{0} = 0$, i.e. at the point $(x, y) = (0, 0)$.

Is $f(x)$ 1-1 ?

Assume $f(c) = f(d)$

$c^3 = d^3$

$\sqrt[3]{c^3} = \sqrt[3]{d^3}$

$c = d$

$\therefore f$ is 1-1

$\therefore f^{-1}$ exists

find $f^{-1}(y)$

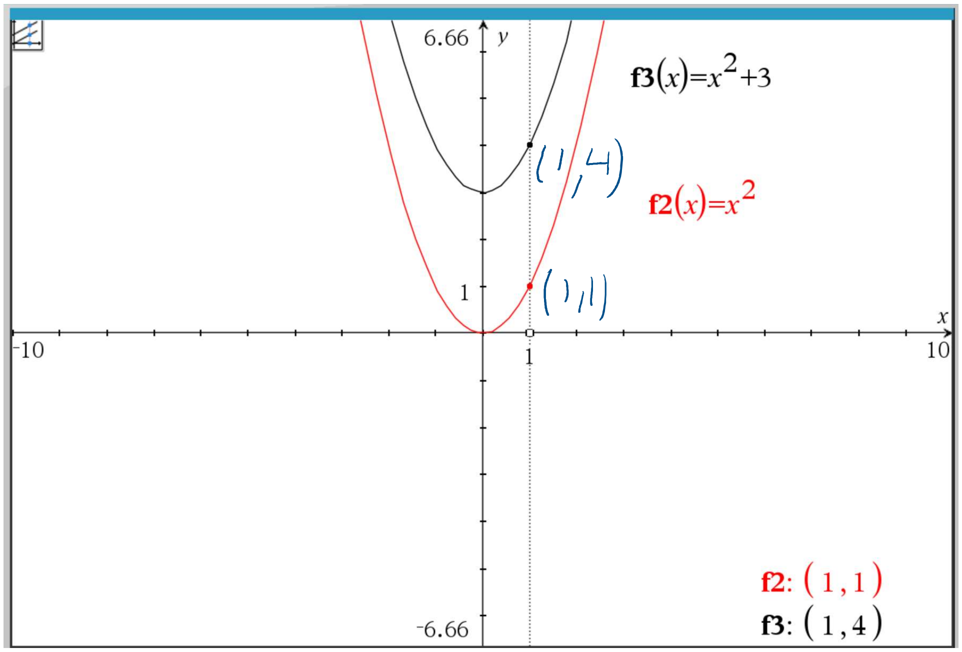
$y = x^3$ switch x, y

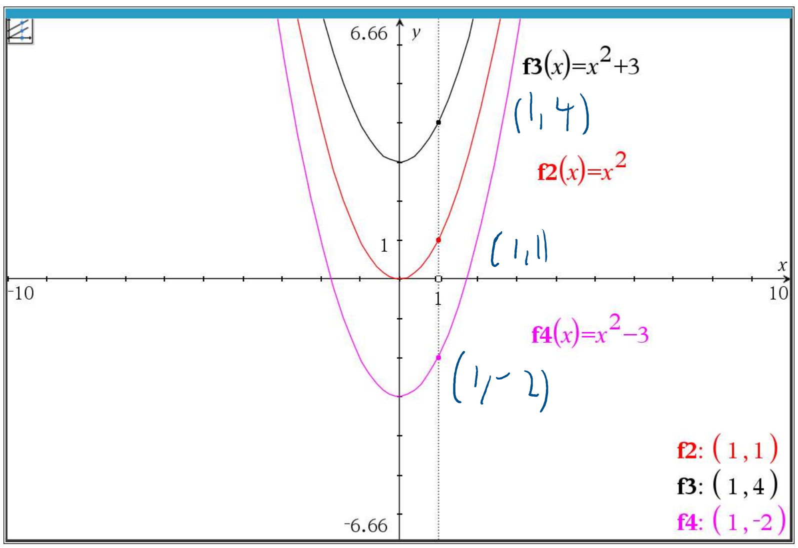
$x = y^3$ solve for y

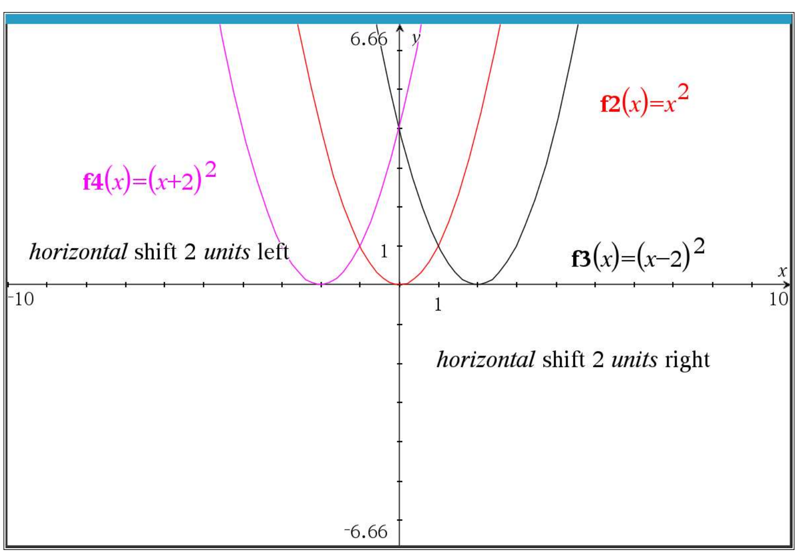
$y = \sqrt[3]{x}$ replace y with $f^{-1}(x)$

$f^{-1}(x) = \sqrt[3]{x}$

$\Rightarrow f^{-1}(y) = \sqrt[3]{y}$


6.66
y
f3(x)=x^2+3
(1,4)
f2(x)=x^2
1
(1,1)
x
-10
1
10
f2: (1,1)
f3: (1,4)
-6.66


6.66
y
f3(x)=x^2+3
(1,4)
f2(x)=x^2
1
(1,1)
x
-10
1
10
f4(x)=x^2−3
(1,-2)
f2: (1,1)
f3: (1,4)
f4: (1,-2)
-6.66


6.66
y
f2(x)=x^2
f4(x)=(x+2)^2
horizontal shift 2 units left
1
f3(x)=(x−2)^2
x
-10
1
10
horizontal shift 2 units right
-6.66

Memorize

Theorem 1.4. Reflections. Suppose f is a function.

- To graph $y = -f(x)$, reflect the graph of $y = f(x)$ across the x-axis by multiplying the y-coordinates of the points on the graph of f by -1.
- To graph $y = f(-x)$, reflect the graph of $y = f(x)$ across the y-axis by multiplying the x-coordinates of the points on the graph of f by -1.

Memorize

Theorem 1.5. Vertical Scalings. Suppose f is a function and $a > 0$. To graph $y = af(x)$, multiply all of the y-coordinates of the points on the graph of f by a. We say the graph of f has been vertically scaled by a factor of a.

- If $a > 1$, we say the graph of f has undergone a vertical stretching (expansion, dilation) by a factor of a.
- If $0 < a < 1$, we say the graph of f has undergone a vertical shrinking (compression, contraction) by a factor of $\frac{1}{a}$.