02-11-25 MTH 263

1.5 The Chain Rule

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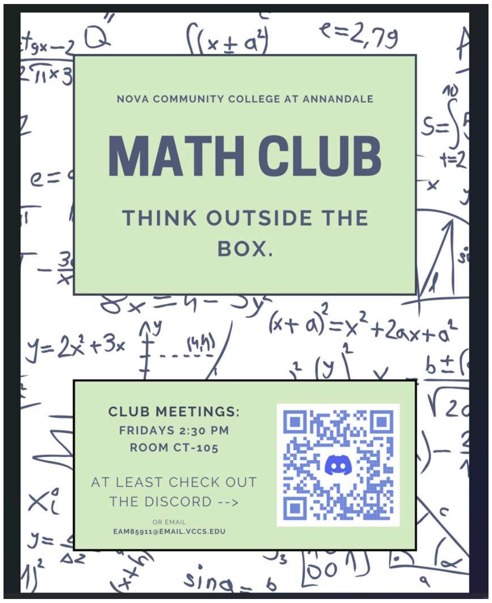
Exam 1, Thursday, 02/13/25

1.1 - 1.5



Office hour notes

1.4:

For Exercises 1-14, use the rules from this section to find the derivative of the given function.

11. $f(x) = \dfrac{ax+b}{cx+d}$ (a, b, c, d are constants)

use quotient rule

$$f'(x) = \frac{(cx+d)\left(\frac{d}{dx}(ax+b)\right) - (ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2}$$

$$= \frac{(cx+d)\left(\frac{d}{dx}(ax) + \frac{d}{dx}(b)\right) - (ax+b)\left(\frac{d}{dx}(cx) + \frac{d}{dx}(d)\right)}{(cx+d)^2}$$

$\frac{d}{dx}(x)$

ok

$$= \frac{(cx+d)\left(a\frac{dx}{dx} + 0\right) - (ax+b)\left(c\frac{dx}{dx} + 0\right)}{(cx+d)^2}$$

$$= \frac{(cx+d)\left((a)(1)\right) - (ax+b)\left((c)(1)\right)}{(cx+d)^2}$$

$(cx+d)(a) - (ax+b)(c$

$(cx+d)$

$$= \frac{(cx+d)(a) - (ax+b)c}{(cx+d)^2}$$

$$= \frac{a\cancel{c}x + ad - \cancel{a}cx - bc}{(cx+d)^2}$$

$$\boxed{f'(x) = \frac{ad-bc}{(cx+d)^2}}$$

1.4: 13

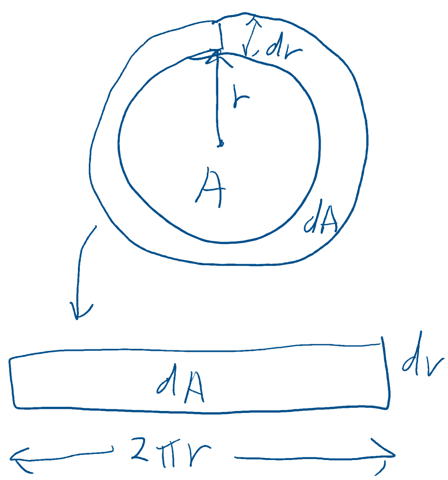
13. $A(r) = \pi r^2$

Find $A'(r)$

$$\boxed{A'(r) = 2\pi r}$$

$$A'(r) = \frac{dA}{dr} = 2\pi r$$

$$dA = 2\pi r \, dr$$



4. $f(x) = \dfrac{\sin x + \cos x}{4}$

$$f'(x) = \frac{d}{dx}\left(\frac{\sin x + \cos x}{4}\right)$$

$$= \left(\frac{1}{4}\right)\frac{d}{dx}(\sin x + \cos x)$$ constant multiple rule

$$= \boxed{\left(\frac{1}{4}\right)(\cos x - \sin x)}$$ sum rule, $\frac{d}{dx}\sin x$, $\frac{d}{dx}\cos x$

1.5: 5

For Exercises 1-18, find the derivative of the given function.

5. $f(x) = \dfrac{\sqrt{x}}{x+1}$ quotient rule

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$$f'(x) = \frac{(x+1)\frac{d}{dx}(\sqrt{x}) - \sqrt{x}\,\frac{d}{dx}(x+1)}{(x+1)^2}$$

$$= \frac{(x+1)\frac{d}{dx}\left(x^{\frac{1}{2}}\right) - \sqrt{x}\,(1+0)}{(x+1)^2}$$

$$= \frac{(x+1)\left(\left(\frac{1}{2}\right)x^{-\frac{1}{2}}\right) - \sqrt{x}}{(x+1)^2}$$

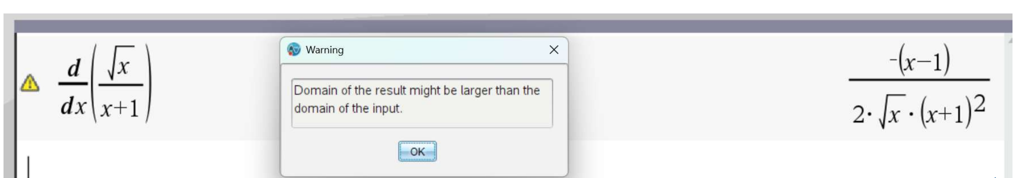
$$= \frac{(x+1)\left(\frac{1}{2\sqrt{x}}\right) - \sqrt{x}}{(x+1)^2}$$

$$= \frac{(x+1)\left(\frac{1}{2\sqrt{x}}\right) - \frac{\sqrt{x}(2\sqrt{x})}{2\sqrt{x}}}{(x+1)^2}$$

$$= \frac{x+1-2x}{2\sqrt{x}\,(x+1)^2}$$

$$= \frac{-x+1}{2\sqrt{x}\,(x+1)^2}$$

$$= \boxed{\frac{(-x+1)\sqrt{x}}{2x(x+1)^2}}$$



domain

$x \neq -1$ to avoid division by 0

$x \geq 0$ to avoid non-real values

$x \neq -1$ to avoid div by 0

$x \neq 0$ to avoid div by 0

$x \geq 0$ to avoid non-real values

$x \geq 0$ to avoid non-real values

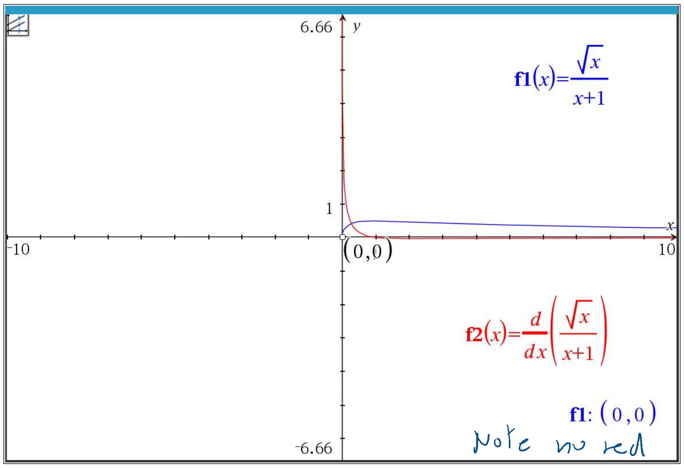
domain $= \{x \mid x \geq 0\}$

$= [0, \infty)$

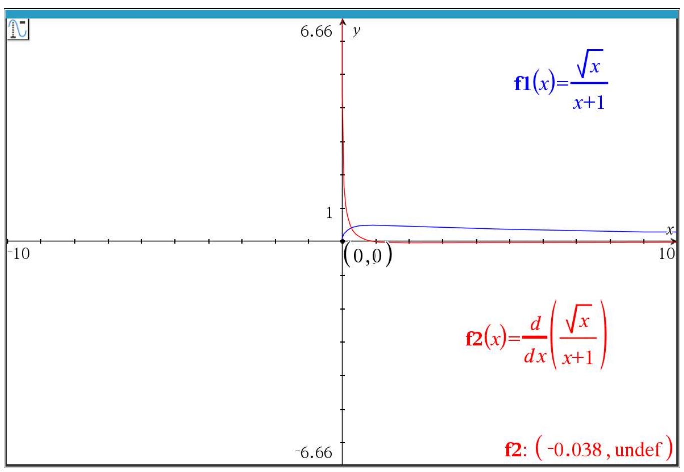
$x > 0$ to avoid non-real values

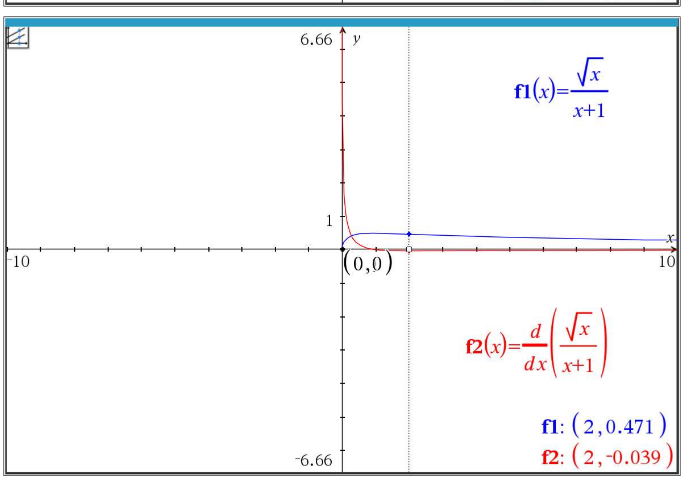
$\{x \mid x > 0\}$

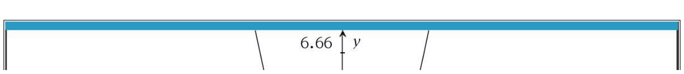
$= (0, \infty)$



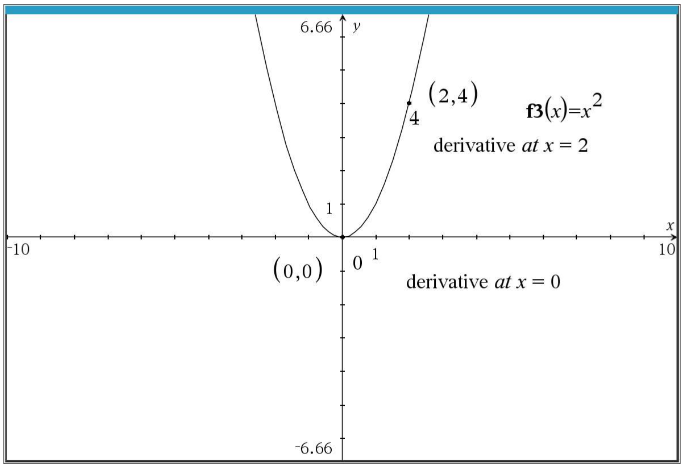
Note no red point here at $x=0$







Calculate the derivative of a function graphically on the TI-84 at a given point



Calculate the derivative of a function graphically on the TI-84 at a given point.

Graph function
2nd Calc
dy/dx
Input a value for x

1.5: 19

B

19. In a certain type of electronic circuit[27] the *overall gain* A_v is given by

$$A_v = \frac{A_o}{1 - T}$$

where the *loop gain* T is a function of the *open-loop gain* A_o.

(a) Show that

$$\frac{dA_v}{dA_o} = \frac{1}{1 - T} - \frac{A_o}{(1 - T)^2}\frac{d(1-T)}{dA_o}.$$

(b) In the case where T is directly proportional to A_o, use part (a) to show that

$$\frac{dA_v}{dA_o} = \frac{1}{(1 - T)^2}.$$

(Hint: First show that $A_o \cdot \frac{d(1-T)}{dA_o} = -T$.)

1.5: 9

For Exercises 1-18, find the derivative of the given function.

9. $f(x) = \sin^2 x = (\sin x)^2$

$f'(x) = 2(\sin x)\frac{d}{dx}(\sin x)$

$= 2 \sin x \cos x$

$= \boxed{\sin(2x)}$

1.2: 5

Note: For all exercises, you can use anything discussed so far (including previous exercises).

For Exercises 1-11, find the derivative of the given function $f(x)$ for all x (unless indicated otherwise).

5. $f(x) = \frac{1}{x+1}$, for all $x \neq -1$

$$f'(x) = \lim_{h \to 0} \frac{\Delta f}{\Delta x}$$

$$\frac{\Delta f}{\Delta x} = \frac{f(x+h) - f(x)}{h}$$

$$= \frac{\frac{1}{(x+h)+1} - \frac{1}{x+1}}{h}$$

$$= \frac{\frac{x+1}{(x+h+1)(x+1)} - \frac{(x+h+1)}{(x+h+1)}\left(\frac{1}{x+1}\right)}{h/1}$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c}$$

$$= \frac{x+1-x-h-1}{h(x+h+1)(x+1)}$$

$$= \frac{-\cancel{h}\,1}{\cancel{h}(x+h+1)(x+1)}$$

$$\boxed{\frac{\Delta f}{\Delta x} = \frac{-1}{(x+h+1)(x+1)}}$$

$$f'(x) = \lim_{h \to 0}\left(\frac{-1}{(x+h+1)(x+1)}\right)$$

$$f'(x) = \frac{-1}{(x+0+1)(x+1)}$$

check

$$f(x) = \frac{1}{x+1} = (x+1)^{-1}$$

$$f'(x) = \frac{\quad}{(x+0+1)(x+1)}$$

$$\boxed{f'(x) = \frac{-1}{(x+1)^2}}$$

$$f(x) = \frac{1}{x+1} = (x+1)$$

extended power rule

$$f'(x) = (-1)(x+1)^{-2}(1)$$

$$f'(x) = (-1)(x+1)^{-2}$$

$$\boxed{f'(x) = \frac{-1}{(x+1)^2}}$$

1.2: 10

10. $f(x)=\sqrt{x^2+1}$

$$f'(x) = \lim_{h \to 0} \frac{\Delta f}{\Delta x}$$

$$\frac{\Delta f}{\Delta x} = \frac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{h}$$

$$(a-b)(a+b) = a^2-b^2$$

$$= \left(\frac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{h}\right)\left(\frac{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}}{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}}\right)$$

$$= \frac{\left[(x+h)^2+1\right] - (x^2+1)}{h\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)}$$

$$= \frac{\cancel{x^2} + 2xh + h^2 \cancel{+1} - \cancel{x^2} \cancel{-1}}{h\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)}$$

$$= \frac{2xh + h^2}{h\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)}$$

$$\overline{h\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)}$$

$$= \frac{\cancel{h}(2x+h)}{\cancel{h}\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)}$$

$$\frac{\Delta f}{\Delta x} = \frac{2x+h}{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}}$$

$$f'(x) = \lim_{h\to 0}\left(\frac{2x+h}{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}}\right)$$

$$= \frac{2x}{\sqrt{x^2+1}+\sqrt{x^2+1}}$$

$$= \frac{2x}{2\sqrt{x^2+1}}$$

$$\boxed{f'(x) = \frac{x}{\sqrt{x^2+1}}}$$

check

$$f(x) = \sqrt{x^2+1} = (x^2+1)^{\frac{1}{2}}$$

$$f'(x) = \left(\tfrac{1}{2}\right)(x^2+1)^{-\frac{1}{2}}(2x)$$

$$\boxed{f'(x) = \frac{x}{\sqrt{x^2+1}}}$$

1.4: 8

For Exercises 1-14, use the rules from this section to find the derivative of the given function.

8. $f(x) = \dfrac{\sin x}{x^2}$

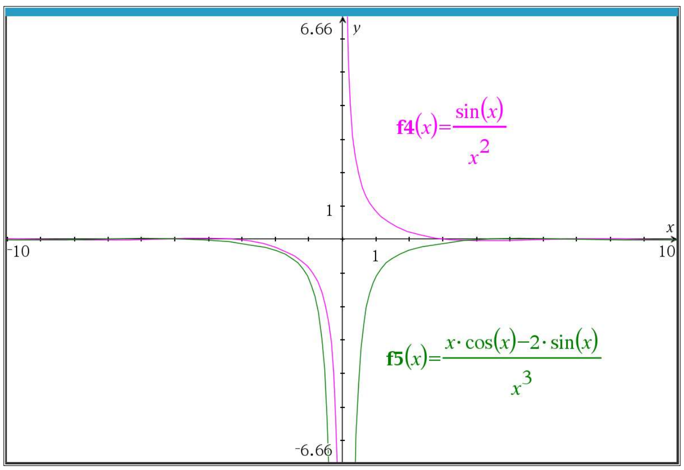
$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

$$f'(x) = \frac{x^2\frac{d}{dx}(\sin x) - (\sin x)\frac{d}{dx}(x^2)}{(x^2)^2}$$

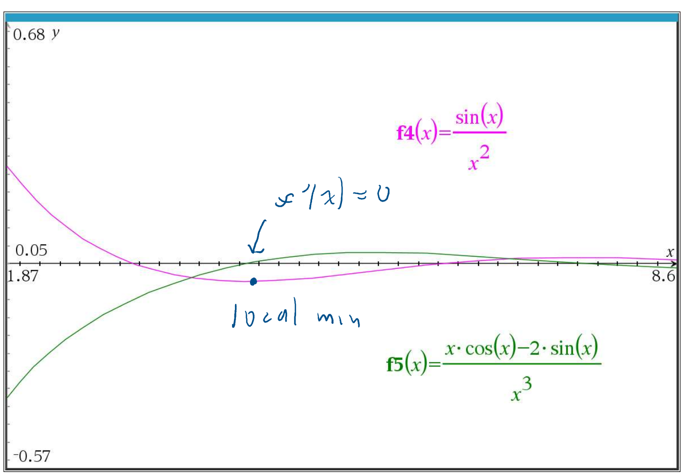
$$(x^2)^2$$

$$= \frac{x^2 \cos x - (\sin x)(2x)}{x^4}$$

$$\boxed{f'(x) = \frac{x \cos x - 2 \sin x}{x^3}}$$



This graph illustrates the fact that the derivative of an odd function is even.



We will study this later.

1.2

For Exercises 1-9, let dx be an infinitesimal and prove the given formula.

1. $(dx + 1)^2 = 2dx + 1$

$(dx+1)^2 = (dx)^2 + 2dx + 1$

1. $(dx + 1)^2 = 2dx + 1$

$$(dx+1)^2 = (dx)^2 + 2dx + 1$$
$$= 0 + 2dx + 1$$
$$= 2dx + 1$$

2. $(dx + 1)^3 = 3dx + 1$

$$(dx+1)^3 = (dx+1)^2(dx+1)$$
$$= (2dx+1)(dx+1)$$
$$= 2(dx)^2 + 2dx + dx + 1$$
$$= 0 + 3dx + 1$$
$$= 3dx + 1$$

Conjecture

$$(dx+1)^n = n\,dx + 1$$

true for $n = 2, 3$

Let $n = 1$, $(dx+1)^1 \overset{?}{=} 1 \cdot dx + 1$

$$dx + 1 = dx + 1$$

Let $n = 0$ $(dx+1)^0 \overset{?}{=} 0 \cdot dx + 1$

$$1 = 1$$

Assume true for any fixed $n \geq 3$

Prove $(dx+1)^{n+1} = (n+1)dx + 1$

Ind hyp $(dx+1)^n = n\,dx + 1$

$$(dx+1)^{n+1} = (dx+1)^n(dx+1) \quad \text{algebra}$$
$$= (n\,dx+1)(dx+1) \quad \text{Ind. hyp}$$

$$= n(dx)^2 + n\,dx + dx + 1$$
$$= (n)(0) + (n+1)\,dx + 1$$
$$= (n+1)\,dx + 1$$

5. $\sin 2dx = 2dx$

$\boxed{\sin dx = dx}$ $\boxed{\cos dx = 1}$

$$\sin 2dx = 2(\sin dx)(\cos dx)$$
$$= 2(dx)(1)$$
$$= 2dx$$

1.5: 5

For Exercises 1-18, find the derivative of the given function.

5. $f(x) = \dfrac{\sqrt{x}}{x+1}$

$$f(x) = \frac{x^{\frac{1}{2}}}{x+1} = (x^{\frac{1}{2}})(x+1)^{-1}$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{\frac{1}{2}}) = \left(\frac{1}{2}\right)x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}(x+1) = 1$$

quotient rule $f'(x) = \dfrac{(x+1)\frac{d}{dx}(x^{\frac{1}{2}}) - x^{\frac{1}{2}}\frac{d}{dx}(x+1)}{(x+1)^2}$

$$= \frac{(x+1)\left(\frac{1}{2\sqrt{x}}\right) - \sqrt{x}\,(1)}{(x+1)^2}$$

$x+1$ $\sqrt{}$

$(x+1)$

$$= \frac{\frac{x+1}{2\sqrt{x}} - \sqrt{x}}{(x+1)^2}$$

$$= \frac{\frac{x+1}{2\sqrt{x}} - \frac{\sqrt{x}(2)\sqrt{x}}{2\sqrt{x}}}{(x+1)^2}$$

$$= \frac{x+1-2x}{(2\sqrt{x})(x+1)^2}$$

$$= \boxed{\frac{1-x}{2\sqrt{x}(x+1)^2}}$$

B

19. In a certain type of electronic circuit[27] the *overall gain* A_v is given by

$$A_v = \frac{A_o}{1-T}$$

where the *loop gain* T is a function of the *open-loop gain* A_o. $T = T(A_0)$

(a) Show that

$$\frac{dA_v}{dA_o} = \frac{1}{1-T} - \frac{A_o}{(1-T)^2}\frac{d(1-T)}{dA_o}.$$

(b) In the case where T is directly proportional to A_o, use part (a) to show that

$T = kA_0$
for some constant k

$$\frac{dA_v}{dA_o} = \frac{1}{(1-T)^2}.$$

(Hint: First show that $A_o \cdot \frac{d(1-T)}{dA_o} = -T$*.)*

(a)

$$A_v = \frac{A_0}{1-T}$$

$$\frac{dA_v}{dA_0} = \frac{(1-T)\frac{d}{dA_0}(A_0) - A_0\frac{d}{dA_0}(1-T)}{(1-T)^2}$$

$$= \frac{(1-T)(1) - A_0\left(\frac{d(1)}{dA_0} - \frac{dT}{dA_0}\right)}{(1-T)^2}$$

$$= \frac{1-T-A_0\left(0-\frac{dT}{dA_0}\right)}{}$$

$$= \frac{1 - T - A_0\left(0 - \frac{dT}{dA_0}\right)}{(1-T)^2}$$

$$= \frac{1 - T + A_0 \frac{dT}{dA_0}}{(1-T)^2}$$

$$= \frac{1-T}{(1-T)^2} + \frac{A_0 \frac{dT}{dA_0}}{(1-T)^2}$$

$$= \frac{1}{1-T} + \frac{A_0 \frac{dT}{dA_0}}{(1-T)^2}$$

$$\boxed{\frac{dA_v}{dA_0} = \frac{1}{1-T} - \frac{A_0 \frac{d}{dA_0}(1-T)}{(1-T)^2}}$$

b) Prove

$$\frac{dA_v}{dA_o} = \frac{1}{(1 - T)^2}.$$

Hint:

(Hint: First show that $A_o \cdot \frac{d(1-T)}{dA_o} = -T$.*)*

If $A_0 \frac{d(1-T)}{dA_0} = -T$, Then $\frac{dA_v}{dA_0} = \frac{1}{1-T} - \frac{(-T)}{(1-T)^2}$

$$= \frac{(1-T) + T}{(1-T)^2}$$

$$= \frac{1}{(1-T)^2}$$

$$(A_0)\frac{d(1-T)}{dA_0}$$

$$= A_0 \frac{d(1)}{dA_0} - A_0 \frac{dT}{dA_0}$$

$$= -A_0 \frac{dT}{dA_0}$$

Let $T = k A_0$, k = constant

$$= -A_0 \frac{d}{dA_0}(k A_0)$$

$$= (-A_0)(k) \frac{dA_0}{dA_0}$$

$$= -A_0 k$$

$$= -T$$

done if $-A_0 \frac{dT}{dA_0} = -T$

$$\Longleftrightarrow \quad A_0 \frac{dT}{dA_0} = T$$

$$\Leftrightarrow \frac{dT}{dA_0} = \frac{T}{A_0}$$

$$\Leftrightarrow \frac{dT}{T} = \frac{dA_0}{A_0}$$

$$\Leftrightarrow \int \frac{dT}{T} = \int \frac{dA_0}{A_0}$$

$$\Leftrightarrow \ln|T| = \ln|A_0| + \ln C = \ln CA_0$$

$$\Leftrightarrow T = e^{\ln A_0 + \ln C}$$

$$\Leftrightarrow \boxed{T = A_0 \cdot e^C}$$ here $K = e^C$

we will study integration later

Let $T = kA_0$