02-04-25 MTH 263

1.4 Derivatives of Sums, Products and Quotients
page 36: 1, 5, 11, 13

1.5 The Chain Rule
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1.3

A

For Exercises 1-9, let dx be an infinitesimal and prove the given formula.

3. $(dx + 1)^{-1} = 1 - dx$

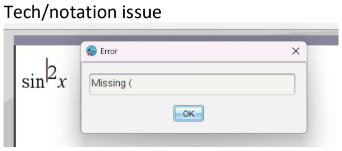
$$(dx+1)^{-1} = \frac{1}{dx+1} = \frac{1}{dx+1}\left(\frac{dx-1}{dx-1}\right)$$

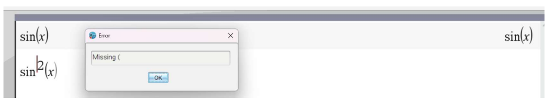
$$= \frac{dx-1}{(dx+1)(dx-1)} = \frac{dx-1}{(dx)^2 - 1^2} = \frac{dx-1}{0-1} = \frac{dx-1}{-1} = 1 - dx$$

$$\frac{1}{dx+1} = \frac{1}{1 + \text{tiny number}} < 1 \Rightarrow \text{our result is plausible}$$

$$1 - dx < 1$$

Tech/notation issue





7. $\sin 3dx = 3dx$

$$\boxed{\begin{aligned}\sin(a+b) &= \sin a \cos b \\ &+ \cos a \sin b\end{aligned}}$$

$$\sin 3dx = \sin(3\,dx)$$

$$\sin(3dx) = \sin(2\,dx + dx)$$

$$= [\sin(2\,dx)][\cos(dx)] + [\cos(2\,dx)](\sin dx)$$

$$= \underbrace{(2\,dx)}_{\#5}\ \underbrace{(1)}_{\text{earlier}} + \underbrace{(1)}_{\#6}\ \underbrace{(dx)}_{\text{earlier}}$$

$$= 2\,dx + dx$$

$$= \boxed{3\,dx}$$

Let $f(x) = x^2$

Find $\frac{df}{dx}$

$$\frac{df}{dx} = \lim_{h \to 0} \frac{\Delta f}{\Delta x} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$= \lim_{h\to 0} (2x + h)$$

use infinitesimals

$$\frac{df}{dx} = \frac{f(x+dx) - f(x)}{dx}$$

$$= \frac{(x+dx)^2 - x^2}{dx}$$

$$= \frac{\cancel{x^2} + 2x\,dx + (dx)^2 - \cancel{x^2}}{dx}$$

$$= \frac{2x\,dx + 0}{dx}$$

$$= \frac{2x\cancel{dx}}{\cancel{dx}}$$

$$= \boxed{2x}$$

$$= 2x + 0$$
$$= \boxed{2x}$$

power rule

$$\frac{d}{dx}(x^2) = 2x^{2-1} = 2x^1 = \boxed{2x}$$

1.1: 2

For Exercises 1-4, suppose that an object moves in a straight line such that its position s after time t is the given function $s = s(t)$. Find the instantaneous velocity of the object at a general time $t \geq 0$. You should mimic the earlier example for the instantaneous velocity when $s = -16t^2 + 100$.

2. $s = 9.8t^2$

$s(t)$ = distance travelled at time $t \geq 0$

$$s(0) = 9.8(0^2) = 0$$
$$s(10) = (9.8)(10^2) = (9.8)(100) = 980$$
$$s(1) = 9.8(1) = 9.8$$
$$s(2) = 9.8(2^2) = (9.8)(4) = 38.2$$

from a later section

$$v = \frac{ds}{dt} = \frac{d}{dt}(9.8t^2) = 9.8\frac{d}{dt}(t^2) = (9.8)(2t) = \boxed{19.6t}$$

Mimicking the earlier example, we find $\frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$, $\Delta t \neq 0$

$$= \lim_{\Delta t \to 0} \frac{s(t+\Delta t) - s(t)}{\Delta t}$$

$$\left[\begin{aligned} &(t+\Delta t) - t \\ &= t + \Delta t - t \\ &= (t - t) + \Delta t \\ &= \Delta t \end{aligned} \right]$$

$$= \lim_{\Delta t \to 0} \frac{9.8(t+\Delta t)^2 - 9.8t^2}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{9.8(t^2 + 2t\Delta t + (\Delta t)^2) - 9.8t^2}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\cancel{9.8t^2} + 19.6t\Delta t + 9.8(\Delta t)^2 - \cancel{9.8t^2}}{\Delta t}$$

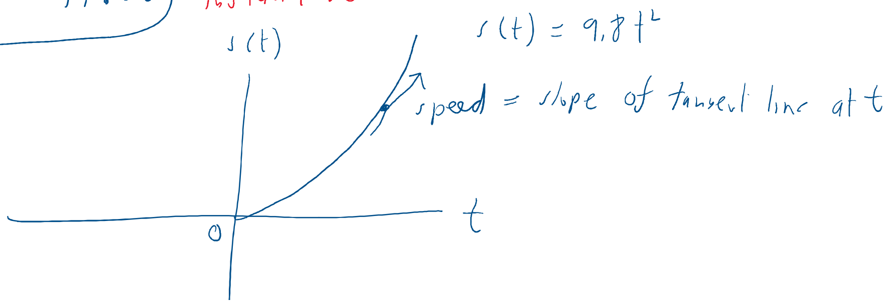
$$= \lim_{\Delta t \to 0} \frac{19.6t\Delta t + 9.8(\Delta t)^2}{\Delta t}$$

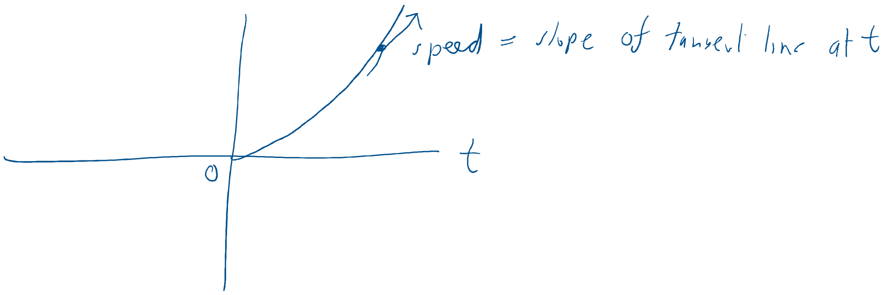
$$= \lim_{\Delta t \to 0} \frac{\cancel{\Delta t}(19.6t + 9.8\Delta t)}{\cancel{\Delta t}}$$

$$= \lim_{\Delta t \to 0} (19.6t + 9.8\Delta t)$$

$$= 19.6t + (9.8)(0)$$

$$\boxed{\frac{ds}{dt} = 19.6t}$$ instantaneous velocity at time $t \geq 0$





1.4: 1

For Exercises 1-14, use the rules from this section to find the derivative of the given function.

1. $f(x) = x^2 - x - 1$

$\boxed{f'(x) = 2x - 1}$ good

long way

$$f'(x) = \frac{d}{dx}\left(x^2 - x - 1\right)$$

$$= \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(-x) + \frac{d}{dx}(-1) \quad \left[\text{sum/difference rule}\right]$$

$$= \underbrace{2x}_{\text{power rule}} - \underbrace{\frac{d}{dx}(x)}_{\text{constant multiple rule}} + \underbrace{0}_{\text{derivative of constant} = 0}$$

$$= 2x - (1)$$

$$= \boxed{2x - 1}$$

Derive the product rule of differentiation using the limit of the difference quotient.

Let $f(x), g(x)$ be differentiable

Find $\frac{d}{dx}\left(f(x) \cdot g(x)\right)$

Let $k(x) = f(x) \cdot g(x)$

Find $\frac{dk}{dx} = \lim_{h \to 0} \frac{k(x+h) - k(x)}{h} = \lim_{h \to 0} \frac{\Delta k}{\Delta x}$

$$\frac{\Delta k}{\Delta x} = \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \frac{f(x+h)g(x+h) + \underbrace{\left[f(x+h)g(x) - f(x+h)g(x)\right]}_{=0} - f(x)g(x)}{h}$$

$$\overline{\qquad\qquad\qquad}$$
$$h$$

$$= \frac{[f(x+h)][g(x+h)-g(x)] + [g(x)[f(x+h)-f(x)]]}{h}$$

$$\frac{\Delta k}{\Delta x} = f(x+h)\left[\frac{g(x+h)-g(x)}{h}\right] + \frac{g(x)[f(x+h)-f(x)]}{h}$$

$$k'(x) = \lim_{h\to 0} \frac{\Delta k}{\Delta x}$$

$$= \lim_{h\to 0} f(x+h)\left[\frac{g(x+h)-g(x)}{h}\right] + \lim_{h\to 0} \frac{g(x)[f(x+h)-f(x)]}{h}$$

$$= \lim_{h\to 0} f(x+h) \cdot \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x) \cdot \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$$

$$= f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

Thm: If f is diff, then f is continuous

Q.E.D.

1.5
The chain rule enables us to differentiate composite functions.

Memorize

Chain Rule: If f is a differentiable function of u, and u is a differentiable function of x, then f is a differentiable function of x, and its derivative with respect to x is:

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

Chain Rule: If g is a differentiable function of x, and f is a differentiable function on the range of g, then $f \circ g$ is a differentiable function of x, and its derivative with respect to x is:

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

$$\frac{d}{dx}\left(x^r\right) = r x^{r-1} \quad \text{for any rational number } r$$

To prove this, let $r = m/n$, where m and n are integers with $n \neq 0$. Then $y = x^r = x^{m/n} = (x^m)^{1/n}$, so that $y^n = x^m$. Taking the derivative with respect to x of both sides of this equation gives

$$\frac{d}{dx}(y^n) = \frac{d}{dx}(x^m)$$, so evaluating the left side by the Chain Rule gives

$$ny^{n-1} \cdot \frac{dy}{dx} = mx^{m-1}$$

$$\left(x^{\frac{m}{n}}\right)^{n-1} = x^{\frac{mn-m}{n}} = x^{m-\frac{m}{n}}$$

$$n\left(x^{m/n}\right)^{n-1} \cdot \frac{dy}{dx} = mx^{m-1}$$

$$\frac{dy}{dx} = \frac{mx^{m-1}}{nx^{m-(m/n)}} = \frac{m}{n}x^{\cancel{m}-1-(\cancel{m}-(m/n))} = \frac{m}{n}x^{(m/n)-1} = rx^{r-1} \quad \checkmark$$

Your Name MTH 263 quiz 2 No calculator.

1. Use the product rule to find $\frac{d}{dx}(x^2 \cdot x^3)$.

$$\frac{d}{dx}(x^2 \cdot x^3) = \frac{d}{dx}(x^5) = \boxed{5x^4}$$

$$= x^2\frac{d}{dx}(x^3) + x^3\frac{d}{dx}(x^2)$$

$$= x^2(3x^2) + x^3(2x)$$

$$= 3x^4 + 2x^4$$

$$= \boxed{5x^4}$$

2. Use the quotient rule to find $\frac{d}{dx}\left(\frac{x^5}{x^3}\right)$.

$$= \frac{d}{dx}(x^2) = \boxed{2x}$$

$$= \frac{x^3\frac{d}{dx}(x^5) - x^5\frac{d}{dx}(x^3)}{(x^3)^2}$$

$$= \frac{x^3(5x^4) - x^5(3x^2)}{x^6}$$

$$= \frac{5x^7 - 3x^7}{x^6} = \frac{2x^7}{x^6} = \boxed{2x}$$

3. Use differentiation rules to find $\frac{df}{dx}$, where

$f(x) = 5x^4 - \sqrt{x} + \frac{1}{x}$.

$$f(x) = 5x^4 - x^{\frac{1}{2}} + x^{-1}$$

$$\frac{df}{dx} = 20x^3 - \left(\frac{1}{2}\right)x^{-\frac{1}{2}} - x^{-2}$$

$| \; ^1/_2 \, x^{-\frac{1}{2}}$ bad notation

$f(x) = 5x^4 - x^{\frac{1}{2}} + x^{-1}$

$$\boxed{\frac{df}{dx} = 20x^3 - \left(\frac{1}{2}\right)x^{-\frac{1}{2}} - x^{-2}}$$

$$\boxed{\frac{df}{dx} = 20x^3 - \frac{1}{2\sqrt{x}} - \frac{1}{x^2}}$$

$1/2\, x^{-\frac{1}{2}}$ bad notation

$\left(\frac{1}{2}\right) x^{-\frac{1}{2}} = \frac{x^{-\frac{1}{2}}}{2}$

4. Let $h(x) = \sin(x^{10})$.
find $\frac{dh}{dx}$.

$$\frac{dh}{dx} = \left[\cos(x^{10})\right](10x^9)$$
$$= 10x^9 \cos(x^{10})$$

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or $u = x^{10}$

$$h(x) = \sin(u(x))$$
$$h'(x) = \frac{d}{du}(\sin(u)) \cdot u'(x)$$
$$= [\cos u][10x^9]$$
$$= [\cos(x^{10})][10x^9]$$
$$= 10x^9 \cos(x^{10})$$

Find the mean of $\{1, 2, 3\}$

$1 + 2 + 3 = 6 \div 3 = \boxed{2}$

$\neq$

$1 + 2 + 3 = 6$

$\frac{6}{3} \div = 2$