01-30-25 MTH 263

1.3 The Derivatives: Infinitesimal Approach
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1.4 Derivatives of Sums, Products and Quotients
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1.3:6

For Exercises 1-9, let dx be an infinitesimal and prove the given formula.

6. $\cos 2dx = 1$ $\quad (dx)^2 = 0$

$\boxed{\sin dx = dx}$

$\boxed{\cos dx = 1}$

$\cos 2dx = 2(\cos dx)$

$\cos(a \cdot b) \stackrel{?}{=} a \cos b$

Let $a = 0$

$\cos(0 \cdot b) \stackrel{?}{=} 0 \cos b$

$\cos(0) \stackrel{?}{=} 0 \cos b$

$1 \neq 0$

$\therefore$ In general $\cos(ab) \neq a \cos b$

Def from linear algebra

T is a linear transformation

if $T(ax + by) = aT(x) + bT(y)$

for any constants a, b

$$\cos(2\,dx) = \cos^2 dx - \sin^2 dx$$
$$= (\cos dx)^2 - (\sin dx)^2$$
$$= 1^2 - (dx)^2$$
$$= 1 - 0$$

$$\boxed{\therefore \cos 2dx = 1}$$

$$\boxed{\therefore \cos 2dx = 1}$$

$$\cos(2dx) = \cos(dx + dx)$$
$$= (\cos dx)(\cos dx) - (\sin dx)(\sin dx)$$
$$= (1)(1) - (dx)(dx)$$
$$= 1$$

13. Show that $\frac{d}{dx}(\cos 2x) = -2\sin 2x$. *(Hint: Use Exercises 5 and 6.)*

$$\frac{d}{dx}(\cos(2x)) = \frac{\cos(2(x+dx)) - \cos(2x)}{dx}$$

$$= \frac{\cos(2x + 2dx) - \cos 2x}{dx}$$

$$= \frac{\cos(2x)\cos(2dx) - \sin(2x)\sin 2dx - \cos 2x}{dx}$$

$$= \frac{\cancel{\cos(2x)}(1) - \sin(2x)(2dx) - \cancel{\cos 2x}}{dx}$$

$$= \frac{-\sin(2x)(2d\cancel{x})}{d\cancel{x}}$$

$$= -2\sin(2x)$$

1.4
Memorize

Rules for Derivatives: Suppose that f and g are differentiable functions of x. Then:

Sum Rule: $\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$

Difference Rule: $\frac{d}{dx}(f-g) = \frac{df}{dx} - \frac{dg}{dx}$

Constant Multiple Rule: $\frac{d}{dx}(cf) = c \cdot \frac{df}{dx}$ for any constant c

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Constant Multiple Rule: $\frac{d}{dx}(cf) = c \cdot \frac{df}{dx}$ for any constant c

Product Rule: $\frac{d}{dx}(f \cdot g) = f \cdot \frac{dg}{dx} + g \cdot \frac{df}{dx}$

Quotient Rule: $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g \cdot \frac{df}{dx} - f \cdot \frac{dg}{dx}}{g^2}$

Sum Rule: $(f+g)'(x) = f'(x) + g'(x)$

Difference Rule: $(f-g)'(x) = f'(x) - g'(x)$

Constant Multiple Rule: $(cf)'(x) = c \cdot f'(x)$ for any constant c

Product Rule: $(f \cdot g)'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$

Quotient Rule: $\left(\frac{f}{g}\right)'(x) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{(g(x))^2}$

$$(f+g)(x) = f(x) + g(x)$$

Derivation of product rule

$$\frac{d}{dx}(f \cdot g) = \frac{(f \cdot g)(x+dx) - (f \cdot g)(x)}{dx}$$

$$d(f \cdot g) = (f \cdot g)(x+dx) - (f \cdot g)(x)$$

Memorize (or be able to derive them quickly)

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$ quotient rule

$$= \cos x \frac{d}{dx}(\sin x) - (\sin x)\frac{d}{dx}(\cos x)$$

$$= \frac{\cos x \frac{d}{dx}(\sin x) - (\sin x)\frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

For $n \geq 1$ differentiable functions $f_1, \ldots, f_n$ and constants $c_1, \ldots, c_n$:

$$\frac{d}{dx}(c_1 f_1 + \cdots + c_n f_n) = c_1 \frac{df_1}{dx} + \cdots + c_n \frac{df_n}{dx} \qquad (1.10)$$

$$\frac{d}{dx}(3\sin x + 4\cos x)$$

$$= \frac{d}{dx}(3\sin x) + \frac{d}{dx}(4\cos x)$$

$$= 3\frac{d}{dx}(\sin x) + 4\frac{d}{dx}(\cos x)$$

$$= 3\cos x - 4\sin x$$

$$\frac{d}{dx}(3\sin x) = 3\frac{d}{dx}(\sin x) = 3\cos x$$

product rule: $3\frac{d}{dx}(\sin x) + (\sin x)\frac{d}{dx}(3)$

$$= 3\cos x + (\sin x)(0)$$

$$= 3\cos x$$

Memorize

Power Rule: $\frac{d}{dx}(x^n) = n x^{n-1}$ for any integer n

$$\frac{d}{dx}\left(x^2\right) = 2 \cdot x^{2-1} = 2 \cdot x^1 = \boxed{2x}$$

$$\frac{d}{dx}\left(x^3\right) = 3 \cdot x^{3-1} = \boxed{3 \cdot x^2}$$

$$\frac{d}{dx}\left(x^{-1}\right) = (-1)x^{-1-1} = (-1)\left(x^{-2}\right) = \boxed{-\frac{1}{x^2}}$$

$$\frac{d}{dx}\left(x^0\right) = 0 \cdot x^{0-1} = 0 \cdot \frac{1}{x} = 0, \text{ for } x \neq 0$$

$$= \frac{d}{dx}(1) = 0$$

Mathematical induction
This is a technique to prove formulas or
propositions that depend on integers.

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

Let $x = 1 + 2 + 3 + \dots + 98 + 99 + 100$

$$x = 100 + 99 + 98 + \dots + \dots + 3 + 2 + 1$$

$$2x = \underbrace{101 + 101 + \dots + 101}_{100 \text{ terms}}$$

$$x = \frac{(100)(101)}{2} = 50(101) = 5050$$

Prove $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for all $n \in \mathbb{Z}^+$

by mathematical induction

Basis step: Let $n = 1$

$$\frac{1}{2} \stackrel{?}{=} 1(1+1)$$

$$\sum_{i=1}^{1} i \stackrel{?}{=} \frac{1(1+1)}{2}$$

$$1 \stackrel{?}{=} \frac{1(2)}{2}$$

$$1 \stackrel{?}{=} \frac{2}{2}$$

$$1 = 1 \quad \checkmark$$

true for $n=1$

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inductive hypothesis

Assume $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for any fixed $n \geq 1$

Prove $\sum_{i=1}^{n+1} i = \frac{(n+1)(n+1+1)}{2}$

$$\sum_{i=1}^{n+1} i = \left(\sum_{i=1}^{n} i\right) + (n+1)$$

$$= \frac{n(n+1)}{2} + (n+1) \quad \text{by ind hyp.}$$

$$= \frac{n(n+1)}{2} + \frac{2(n+1)}{2}$$

$$= \left(\frac{n+1}{2}\right)(n+2) \quad \dashv$$

Prove $2^n > n^2$ for all $n \geq 5$

basis step $2^0 \stackrel{?}{>} 0^2$

let $n=0$

$1 > 0$ true

Let $n=0$ $\quad 1 > 0$ true

Let $n=1$ $\quad 2^1 \overset{?}{>} 1^2$

$2 > 1$ true

Let $n=2$ $\quad 2^2 \overset{?}{>} 2^2$

$4 = 4$ False

Let $n=3$ $\quad 2^3 \overset{?}{>} 3^2$

$8 > 9$ False

Let $n=4$ $\quad 2^4 \overset{?}{>} 4^2$

$16 > 16$ False

Let $n=5$ $\quad 2^5 \overset{?}{>} 5^2$

$32 > 25$ True

Assume $2^n > n^2$ for any fixed $n \geq 5$

Prove $2^{n+1} > (n+1)^2$

$2^{n+1} = 2^n \cdot 2 > (n^2)(2)$ by inductive hypothesis

we would be done if $2n^2 > (n+1)^2$

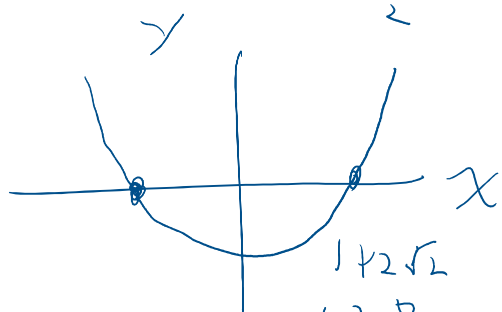
$\Leftarrow$ $2n^2 > n^2 + 2n + 1$

$\Leftarrow$ $n^2 - 2n - 1 > 0$ by below, true for $n \geq 4$

Find zeros of $x^2 - 2x - 1 = 0$

$$x = \frac{2 \pm \sqrt{4 - 4(-1)}}{2}$$

$$x = \frac{2 \pm 4\sqrt{2}}{2} = 1 \pm 2\sqrt{2}$$



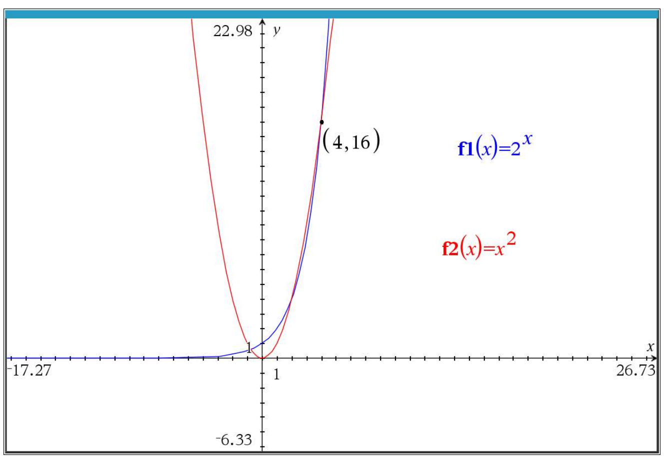
$1 + 2\sqrt{2}$
$\approx 3,8$

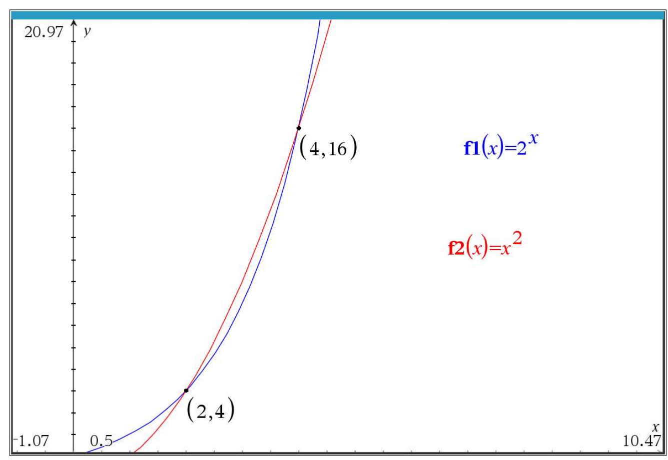
we only consider $x > 0$

$\therefore$ $x^2 - 2x - 1 > 0$
for $x > 1 + \sqrt{2}$

$n^2 - 2n - 1 > 0$
for $n > 3$

since we required
$n \geq 5$ from basis step
$n^2 - 2n - 1 > 0$





$$\boxed{\therefore\ 2^n > n^2 \text{ for all } n \geq 5}$$