01-28-25 MTH 263

1 The Derivative

1.1 Introduction

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1.2 The Derivative: Limit Approach

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1.3 The Derivatives: Infinitesimal Approach

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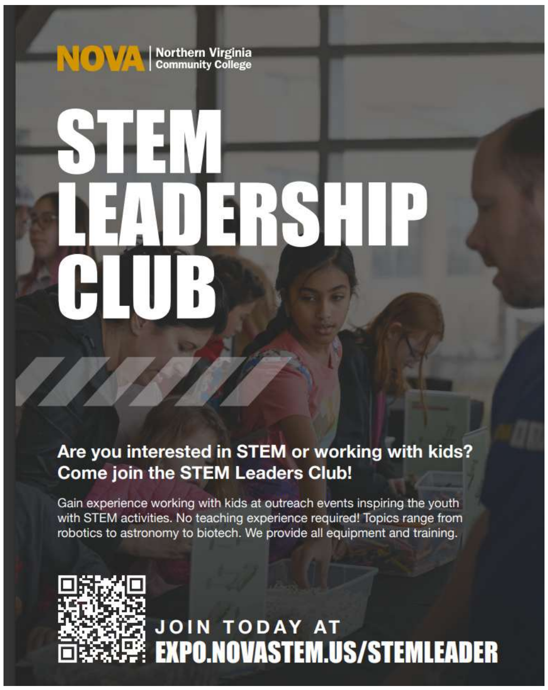


Your Name MTH 263 quiz 1 Write each problem. Show calculations. Put a box around each answer.

1. Find the limit or show that it does not exist.

$$\lim_{x\to 1} \frac{x^2-1}{x-1}$$

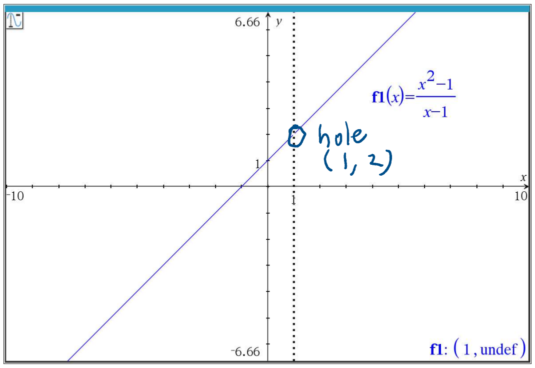
$$= \lim_{x\to 1} \frac{(x+1)(x-1)}{x-1}$$

$$= \lim_{x\to 1} (x+1) = 1+1 = \boxed{2}$$

$\frac{x^2-1}{x-1} = x+1$ for $x \neq 1$

$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1}(x+1)$

$= 1+1 = 2$



2. Let $f(x) = 4x - 3$,

Find $f'(x) = \frac{df}{dx}$ using the difference quotient.

$$\frac{\Delta f}{\Delta x} = \frac{f(x+h) - f(x)}{h}$$

$$= \frac{[4(x+h)-3] - [4x-3]}{h}$$

$$= \frac{\cancel{4x} + 4h \cancel{-3} - \cancel{4x} \cancel{+3}}{h}$$

$$= \frac{4\cancel{h}}{\cancel{h}}$$

$$\boxed{\frac{\Delta f}{\Delta x} = 4}$$

$$f'(x) = \lim_{h \to 0} 4 = \boxed{4}$$

1.3

Memorize

Notation for the derivative of $y = f(x)$: The following are all equivalent:

$$\frac{dy}{dx}\ ,\ f'(x)\ ,\ \frac{d}{dx}(f(x))\ ,\ y'\ ,\ \dot{y}\ ,\ \dot{f}(x)\ ,\ \frac{df}{dx}\ ,\ Df(x)$$

Supplied

A number δ is an **infinitesimal** if the conditions (a)-(d) hold:

(a) $\delta \neq 0$

(b) if $\delta > 0$ then δ is smaller than any positive real number

(c) if $\delta < 0$ then δ is larger than any negative real number

(d) $\delta^2 = 0$ (and hence all higher powers of δ, such as δ^3 and δ^4, are also 0)

Note: Any infinitesimal multiplied by a nonzero real number is also an infinitesimal, while 0 times an infinitesimal is 0.

Supplied

Let dx be an infinitesimal such that $f(x+dx)$ is defined. Then $dy = f(x+dx) - f(x)$ is also an infinitesimal, and the derivative of $y = f(x)$ at x is the ratio of dy to dx:

$$\frac{dy}{dx} = \frac{f(x+dx) - f(x)}{dx} \tag{1.8}$$

Let $y = f(x) = 4x - 3$

$$f'(x) = \frac{dy}{dx} = \frac{f(x+dx) - f(x)}{dx}$$

$$= \frac{[4(x+dx) - 3] - [4x - 3]}{dx}$$

$$= \frac{\cancel{4x} + 4dx \cancel{- 3} - \cancel{4x} + \cancel{3}}{dx}$$

$$= 4\frac{\cancel{dx}}{\cancel{dx}} = \boxed{4}$$

Let $y = f(x) = x^2$

$$f'(x) = \frac{dy}{dx} = \frac{f(x+dx) - f(x)}{dx}$$

$$= \frac{(x+dx)^2 - x^2}{dx} \qquad [\text{Note } (dx)^2 = 0]$$

$$= \frac{[\cancel{x^2} + 2x\,dx + (dx)^2] - \cancel{x^2}}{dx}$$

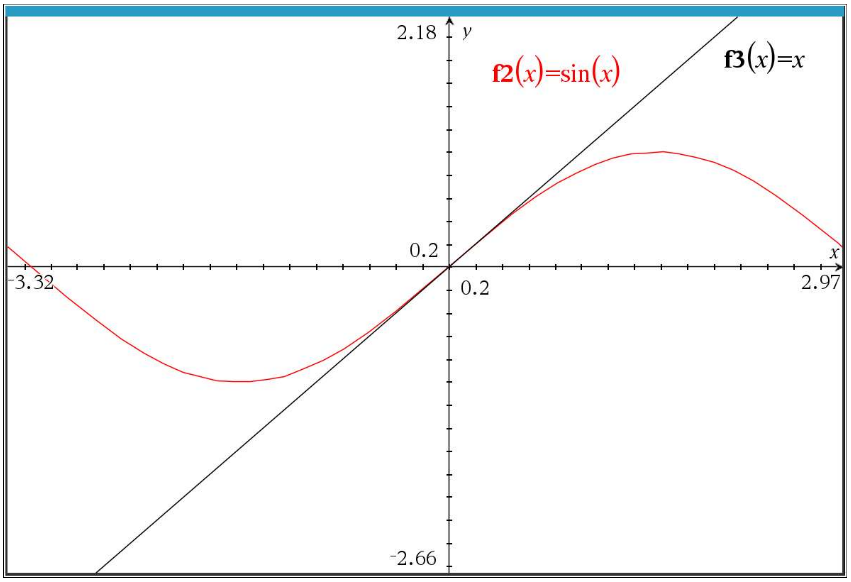
$$= \frac{2x\,dx + 0}{dx}$$

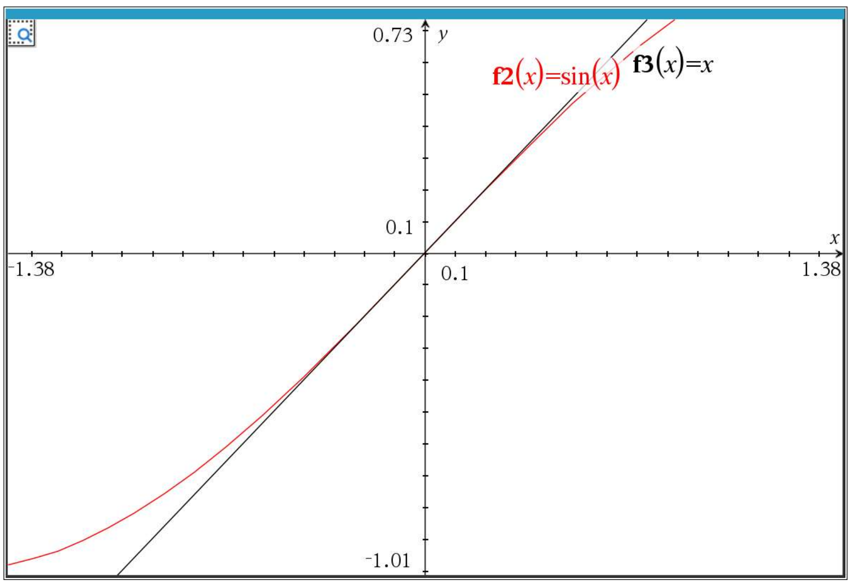
$$= 2x\frac{\cancel{dx}}{\cancel{dx}}$$

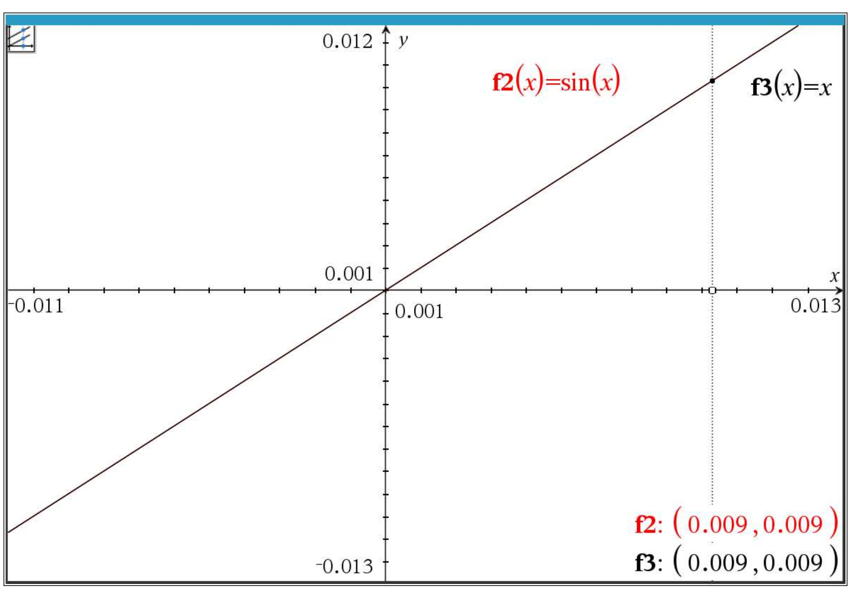
$$= \boxed{2x}$$

Supplied

Microstraightness Property: For the graph of a differentiable function, any part of the curve with infinitesimal length is a straight line segment.


2.18
y
f2(x)=sin(x)
f3(x)=x
0.2
x
-3.32
0.2
2.97
-2.66


0.73
y
f2(x)=sin(x)
f3(x)=x
0.1
x
-1.38
0.1
1.38
-1.01


0.012
y
f2(x)=sin(x)
f3(x)=x
0.001
x
-0.011
0.001
0.013
f2: (0.009 , 0.009)
f3: (0.009 , 0.009)
-0.013

For a differentiable function $f(x)$, $\frac{df}{dx} = f'(x)$ and so multiplying both sides by dx yields the important relation:

$$df = f'(x)\,dx \tag{1.9}$$

1.3: 4

A

For Exercises 1-9, let dx be an infinitesimal and prove the given formula.

4. $\tan dx = dx$

$$\tan dx = \frac{\sin dx}{\cos dx} = \frac{dx}{1} = dx$$

Q.E.D. Latin
Quod erat demonstrandum
"That which was to be proved"

$\boxed{\sin dx = dx}$

$\boxed{\cos dx = 1}$

C

14. Show that $\frac{d}{dx}(\tan x) = \sec^2 x$. *(Hint: Use Exercise 4.)*

- $\tan(\alpha \pm \beta) = \dfrac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$

$$\frac{d}{dx}(\tan x) = \frac{\tan(x + dx) - \tan x}{dx}$$

$$= \frac{\dfrac{\tan x + \tan dx}{1 - \tan x \tan dx} - \tan x}{dx}$$

$$= \frac{\dfrac{\tan x + dx}{1 - (\tan x)dx} - \tan x}{dx}$$

$$= \frac{\dfrac{\tan x + dx}{1 - (\tan x)dx}\,\dfrac{1 + (\tan x)dx}{1 + (\tan x)dx} - \tan x}{dx}$$

$$= \left(\frac{1}{dx}\right)\left(\frac{\tan x + dx + (\tan^2 x)dx + (\tan x)(dx)^2}{1 - (\tan^2 x)(dx)^2} - \tan x\right)$$

$$= \left(\frac{1}{dx}\right)\left(\frac{\tan x + dx + (\tan^2 x)dx + 0}{1 - (\tan^2 x)(0)} - \tan x\right)$$

$$= \left(\frac{1}{dx}\right)\left(\frac{\tan x + dx + (\tan^2 x)dx}{1} - \tan x\right)$$

$$= \left(\frac{1}{dx}\right)\ (dx + (\tan^2 x)dx)$$

$$= 1 + \tan^2 x$$

$$\boxed{\frac{d}{dx}(\tan x) = \sec^2 x}$$