01-23-25 MTH 263

1 The Derivative
- 1.1 Introduction
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division by zero

$\frac{1}{0}$ undef

Assume $\frac{1}{0} = x$ for some real number x

$$\Rightarrow \quad 1 = 0 \cdot x$$
implies

$$1 = 0$$
$$1+1 = 0+1$$
$$2 = 1 = 0$$
$$\vdots$$
$$1{,}000{,}000 = 0$$

$$\left.\begin{array}{cccc} 2, & 4, & 6, & 8, \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 1, & 2, & 3, & 4, \ldots \end{array}\right\} \text{ same cardinality}$$

count the rational numbers

$$\begin{array}{cccccc} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \ldots \\ \frac{2}{1} & \cancel{\frac{2}{2}} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \ldots \\ \frac{3}{1} & & & & & \\ \vdots & & & & & \end{array}$$

$$\frac{p}{q}$$

$a_1 = 1$

$a_2 = \frac{1}{2}$

$a_3 = 2$

$a_4 = 3$

$a_5 = \frac{1}{3}$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots ,$$

pi/4=0.785398163397448
1-1/3=0.6667
1-1/3+1/5=0.8667

1-1/3+1/5-1/7=0.7238
1-1/3+1/5-1/7+1/9=0.8349
1-1/3+1/5-1/7+1/9-1/11=0.744
1-1/3+1/5-1/7+1/9-1/11+1/13=0.8209
1-1/3+1/5-1/7+1/9-1/11+1/13-1/15=0.7543

$$\sum_{j=1}^{5} \frac{1}{2^j} = \frac{31}{32} = 0.9687$$

$$\sum_{j=1}^{10} \frac{1}{2^j} = 0.99902$$

$$= 1.0$$

$$\sum_{j=1}^{\infty} \frac{1}{2^j} = 1$$

memorize

$\mathbb{N}$ = the set of all **natural** numbers, i.e. the set of nonnegative integers: $0, 1, 2, 3, 4, \ldots$

$\mathbb{Z}$ = the set of all integers: $0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots$

$\mathbb{Q}$ = the set of all **rational** numbers $\frac{m}{n}$, where m and n are integers, with $n \neq 0$

$\mathbb{R}$ = the set of all real numbers

Note that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

$0.999\ldots = 1$

Let $x = 0.999\ldots$

$10x = 9.999\ldots$

$$\begin{array}{rcl} 10x & & 9.999\ldots \\ -\ x & & -0.999\ldots \\ \hline 9x & = & 9 \end{array}$$

$x = \frac{9}{9} = 1$

1.2 memorize

The **derivative** of a real-valued function $f(x)$, denoted by $f'(x)$, is

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \qquad (1.3)$$

for x in the domain of f, provided that the limit exists.[11]

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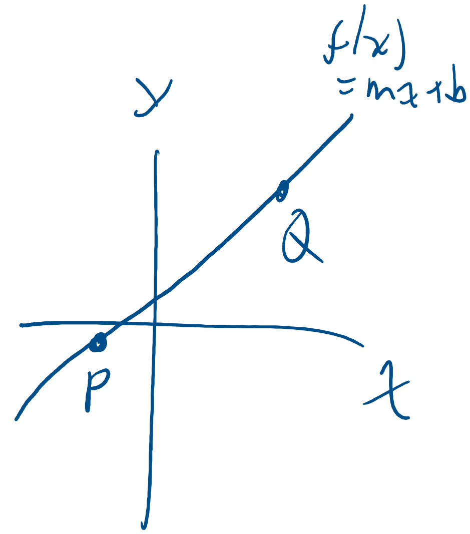
Let $f(x) = mx + b$, m, b constants



$$\frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \frac{[m(x+\Delta x)+b] - [mx+b]}{\Delta x}$$

$$= \frac{\cancel{mx} + m\Delta x + \cancel{b} - \cancel{mx} - \cancel{b}}{\Delta x}$$

$$= \frac{m\cancel{\Delta x}}{\cancel{\Delta x}} = m$$

$$\boxed{\frac{\Delta f}{\Delta x} = m}$$

$$f'(x) = \lim_{\Delta x \to 0} (m) = \boxed{m}$$

Let $f(x) = x^2$

Find $f'(x)$

$$\frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$= \frac{\cancel{x^2} + 2x\Delta x + (\Delta x)^2 - \cancel{x^2}}{\Delta x}$$

$$= \frac{2x\Delta x + (\Delta x)^2}{\Delta x}$$

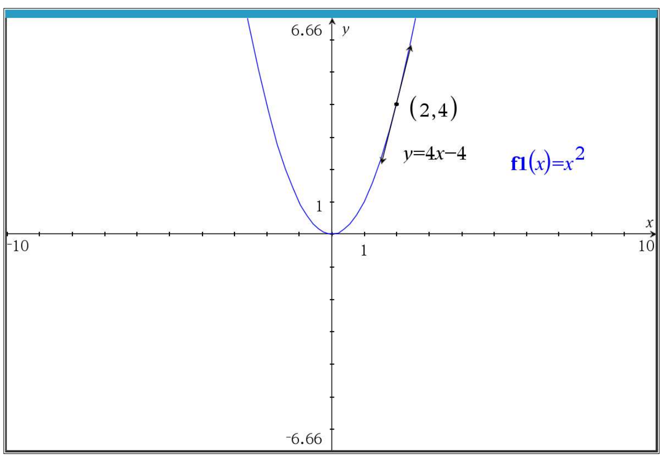
$$\overline{\quad\quad\quad}\over{\Delta x}$$

$$= \frac{\cancel{\Delta x}(2x+\Delta x)}{\cancel{\Delta x}}$$

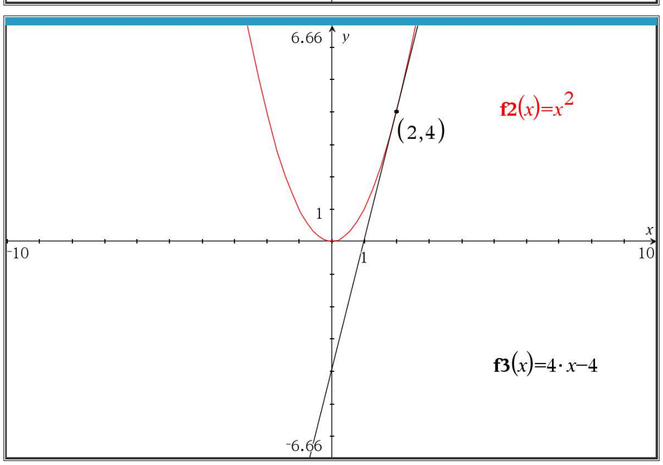
$$\boxed{\frac{\Delta f}{\Delta x} = 2x + \Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x + 0 = 2x$$

$$\boxed{f'(x) = 2x}$$



Draw tangent line at $x = 2$



graph $f(x) = x^2$ and $y = 4x - 4$ find intersection

informal limit definition

For a real number a and a real-valued function $f(x)$, say that the *limit* of $f(x)$ as x approaches a equals the number L, written as

$$\lim_{x\to a} f(x) = L\,,$$

if $f(x)$ approaches L as x approaches a.
Equivalently, this means that $f(x)$ can be made as close as you want to L by choosing x close enough to a. Note that x can approach a from any direction.

memorize

Rules for Limits: Suppose that a is a real number and that $f(x)$ and $g(x)$ are real-valued functions such that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist. Then:

(a) $\displaystyle\lim_{x\to a}(f(x)+g(x)) = \left(\lim_{x\to a} f(x)\right) + \left(\lim_{x\to a} g(x)\right)$

(b) $\displaystyle\lim_{x\to a}(f(x)-g(x)) = \left(\lim_{x\to a} f(x)\right) - \left(\lim_{x\to a} g(x)\right)$

(c) $\displaystyle\lim_{x\to a}(k\cdot f(x)) = k\cdot\left(\lim_{x\to a} f(x)\right)$ for any constant k

(d) $\displaystyle\lim_{x\to a}(f(x)\cdot g(x)) = \left(\lim_{x\to a} f(x)\right)\cdot\left(\lim_{x\to a} g(x)\right)$

(e) $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, if $\displaystyle\lim_{x\to a} g(x) \neq 0$

Another formulation is to set $h = w - x$ in formula (1.4), which yields

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{w-x\to 0}\frac{f(x+(w-x)) - f(x)}{w - x}\,,$$

so that

$$\boxed{f'(x) = \lim_{w\to x}\frac{f(w) - f(x)}{w - x}}$$

supplied