

Please give me an integration problem in which it is easier to find an area by integrating with respect to y . Do not give me a hint or a solution.

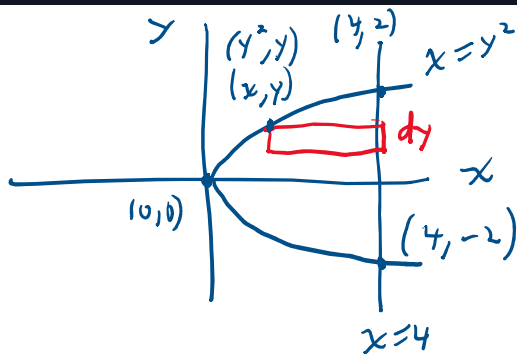
Here's a problem for you:

Problem:

Find the area of the region bounded by the curves

$$x = y^2 \quad \text{and} \quad x = 4.$$

This is a case where integrating with respect to y is more natural.



length of typical rectangular slice
 $= \text{right} - \text{left}$
 $= 4 - y^2$

$$A_{\text{region}} = \int_{-2}^2 (4 - y^2) dy$$

$$= \left(4y - \frac{y^3}{3} \right) \Big|_{-2}^2$$

$$= \left(8 - \frac{8}{3} \right) - \left(-8 - \left(-\frac{8}{3} \right) \right)$$

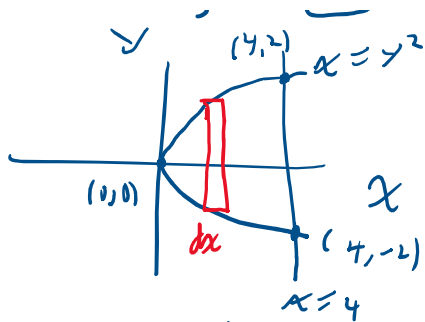
$$= 8 - \frac{8}{3} + 8 - \frac{8}{3}$$

$$= 16 - \frac{16}{3}$$

$$= \frac{48 - 16}{3} = \frac{32}{3}$$

$$32/3 = 10.6667$$





$$\text{Area} = \int_0^4 (\text{top} - \text{bottom}) dx$$

$$x = y^2$$

$$\Rightarrow y = \pm\sqrt{x}$$

$$\text{Area} = \int_0^4 (\sqrt{x} - (-\sqrt{x})) dx$$

$$= \int_0^4 2\sqrt{x} dx$$

$$= 2 \int_0^4 x^{\frac{1}{2}} dx$$

$$= 2 \left[\frac{2x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^4$$

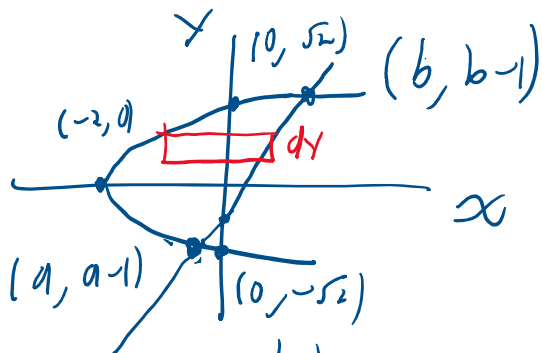
$$= \frac{4}{3} \left[4^{\frac{3}{2}} - 0^{\frac{3}{2}} \right]$$

$$= \left(\frac{4}{3} \right) (8 - 0)$$

$$= \boxed{\frac{32}{3}}$$

Okay, I like the example you just gave me. Could you just give me another example which is a little more difficult, but don't give me the solution?

Absolutely! Here's another one for you: consider the region enclosed by the parabola $x = y^2 - 2$ and the line $y = x - 1$. It's a bit more challenging and you'll likely find it easier to integrate with respect to y . Give it a shot!



$$\text{Area} = \int_{a-1}^{b+1} (\text{right} - \text{left}) dy$$

$$= \int_{a-1}^{b+1} [(y+1) - (y^2-2)] dy$$

$$x = y^2 - 2$$

$$y = x - 1$$

$$(x-1)^2 - 2 = x$$

$$x^2 - 2x + 1 - 2 = x$$

$$x^2 - 3x - 1 = 0$$

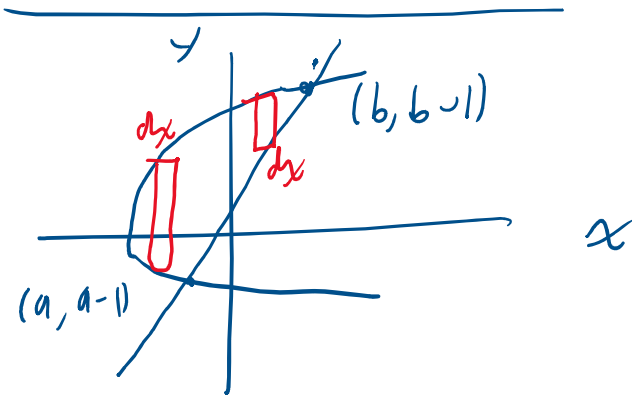
$$x = \frac{3 \pm \sqrt{9 - 4(-1)}}{2}$$

$$x = \frac{3 \pm \sqrt{13}}{2}$$

$$x = a, b$$

$$y = x - 1 \Rightarrow x \Rightarrow y + 1$$

$$x = y^2 - 2$$



Informal definition of limit

$$\lim_{x \rightarrow c} f(x) = L$$

As x approaches c , $f(x)$ approaches L

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that}$$

$$\uparrow \quad \quad \uparrow$$

$$\text{for } \quad \quad \text{there}$$

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$$

↑
for
any

↑
there
exists

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$$

↑
implies

Find $\lim_{x \rightarrow 1} (x + 3)$

verify using the formal definition of limit

$$\lim_{x \rightarrow 1} (x + 3) = 1 + 3 = \boxed{4}$$

let $\epsilon > 0$. Find $\delta > 0$ such that

$$0 < |x - 1| < \delta \Rightarrow |(x + 3) - 4| < \epsilon$$

$$|(x + 3) - 4| < \epsilon$$

$$\Leftrightarrow |x - 1| < \epsilon$$

$$\text{Let } \delta = \epsilon$$

differentials

$$f'(x) = \frac{df}{dx}$$

$$df = f'(x) dx$$

standard calculus

$$f'(x) \Delta x \approx f'(x) dx = df$$

Copilot

Using non-standard analysis, can you give a simple problem using infinitesimals for calculus I students?

Absolutely! Here's a **student-friendly problem** framed in the language of **non-standard analysis** with infinitesimals, designed for a Calculus I audience. It keeps the rigor light but shows the power of infinitesimals in a way they can grasp.



Problem: Slope of a Parabola Using Infinitesimals

Let $f(x) = x^2$.

We want to find the slope of the curve at $x = 3$ using **infinitesimals**.

Step 1: Introduce an infinitesimal

Take an infinitesimal number ε (a positive number smaller than any real number, but not zero).

Look at the change in $f(x)$ when we move from $x = 3$ to $x = 3 + \varepsilon$.

$$\Delta y = f(3 + \varepsilon) - f(3)$$

Step 2: Compute the difference

$$\Delta y = (3 + \varepsilon)^2 - 3^2 = 9 + 6\varepsilon + \varepsilon^2 - 9 = 6\varepsilon + \varepsilon^2$$

Step 3: Form the "infinitesimal slope"

$$\frac{\Delta y}{\Delta x} = \frac{6\varepsilon + \varepsilon^2}{\varepsilon} = 6 + \varepsilon$$

Step 4: Take the standard part

The slope is the **real number closest to** $6 + \varepsilon$.

Since ε is infinitesimal, its standard part is 0.

$$\text{slope at } x = 3 = \text{st}(6 + \varepsilon) = 6$$

🌟 Final Answer

Using infinitesimals, the slope of $y = x^2$ at $x = 3$ is **6**.

Why this works well for Calc I students

- It mirrors the usual derivative process but avoids limits.
- Students see how infinitesimals “cancel out” naturally.
- It reinforces the idea of the derivative as a slope without epsilon-delta formalism.
- It's short, concrete, and connects directly to their familiar derivative rules.

This is a bit from our textbook approach.

Use infinitesimals to find $\frac{dy}{dx}$ for $y = x^2$ at $x = 3$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x+dx)^2 - x^2}{dx} \\ &= \frac{\cancel{x^2} + 2x dx + (dx)^2 - \cancel{x^2}}{dx} \\ &= \frac{2x dx + 0}{dx} \\ &= \frac{2x \cancel{dx}}{\cancel{dx}} \\ &= 2x\end{aligned}$$

$\delta^2 = 0$
for $\delta = \text{infinitesimal}$

$$\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx}(3) = 2(3) = 6$$

Copilot

OK, please give me a u-substitution problem for calculus without giving me a hint or a solution.

Sure, I've got a neat one for you. Let's say you want to integrate the function e raised to the power of $2x$. So your integral is the integral of $(e \text{ to the } 2x) dx$. Now, you're on your own from here—no hints!

Then, integrate from $x=0$ to $x=1$

$$I = \int e^{2x} dx$$

$$\begin{aligned} u &= 2x \\ \Rightarrow du &= 2 dx \\ \Rightarrow dx &= \frac{du}{2} \end{aligned}$$

$$I = \frac{1}{2} \int e^u du$$

$$I = \frac{1}{2} e^u$$

$$I = \left(\frac{1}{2} \right) e^{2x} + C$$

$$\begin{aligned} u &= e^{2x} \\ \Rightarrow du &= 2e^{2x} dx \\ e^{2x} dx &= \frac{du}{2} \end{aligned}$$

$$I = \int \frac{du}{2}$$

$$= \frac{1}{2} (u)$$

$$I = \frac{1}{2} (e^{2x}) + C$$

$$I = \int_0^1 e^{2x} dx$$

$$I = \int_0^1 e^{2x} dx$$

$$u = 2x$$

$$\Rightarrow du = 2 dx$$

$$\Rightarrow dx = \frac{du}{2}$$

$$x=0 \Rightarrow u = (2)(0) = 0$$

$$x=1 \Rightarrow u = (2)(1) = 2$$

$$I = \frac{1}{2} \int_0^2 e^u du$$

$$= \frac{1}{2} [e^u]_0^2$$

$$= \frac{1}{2} [e^2 - e^0]$$

$$\boxed{I = \frac{1}{2} [e^2 - 1]}$$

$$\int_0^1 e^{2 \cdot x} dx$$

$$\frac{e^2}{2} - \frac{1}{2}$$

$$\left(\frac{e^2}{2} - \frac{1}{2} \right) \rightarrow \text{Decimal}$$

$$3.19453$$