12-01-25 MTH 263

Review for final exam

8.2: 10

10. Electrical signals are commonly represented by a *periodic waveform* $x(t)$, which is a function of time t and has period T (i.e. T is the smallest positive number such that $x(t+T) = x(t)$ for all t). The *average power* of the waveform is defined as the average value of its square over a single period:

$$\langle x^2(t) \rangle = \frac{1}{T} \int_0^T x^2(t)\, dt\ .$$

(a) Find the average power of the waveform $x(t) = A\cos(\omega t + \phi)$, where $A > 0$ and $\omega > 0$ and ϕ are all constants.

(b) The *root mean square* of a waveform, abbreviated as *rms*, is the square root of the average power. Calculate the rms of the waveform from part (a). Write your answer in decimal form as a percentage of the amplitude A.

(a) $\langle x^2(t) \rangle = \frac{1}{T}\int_0^T A^2 \cos^2(wt+\phi)\, dt$

$= \frac{A^2}{T}\int_0^T \cos^2(wt+\phi)\, dt$

derive half-angle formula

$\sin^2\theta + \cos^2\theta = 1$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$\Rightarrow \cos^2\theta = \cos(2\theta) + \sin^2\theta$

$\cos^2\theta = \cos(2\theta) + (1 - \cos^2\theta)$

$\cos(2\theta) = \cos^2\theta - 1 + \cos^2\theta$

$\cos(2\theta) = 2\cos^2\theta - 1$

$\boxed{\cos^2\theta = \frac{1+\cos(2\theta)}{2}}$

$\langle x^2(t) \rangle = \frac{A^2}{T}\int_0^T \frac{1+\cos(2wt+2\phi)}{2}\, dt$

$= \frac{A^2}{T}\int_0^T \frac{dt}{2} + \underbrace{\frac{A^2}{T}\int_0^T \cos(2wt+2\phi)\, dt}_{I}$

$I = 0$ since cos is integrated over 1 period T

$= \frac{A^2}{2T}\left[t\right]_0^T$

$= \frac{A^2}{2T}\left[T-0\right]$

$= \boxed{\frac{A^2}{2}}$

Let $u = 2wt + 2\phi$

$\Rightarrow du = 2w\, dt$

$\Rightarrow dt = \frac{du}{2w}$

$t = 0 \Rightarrow u = 2\phi$

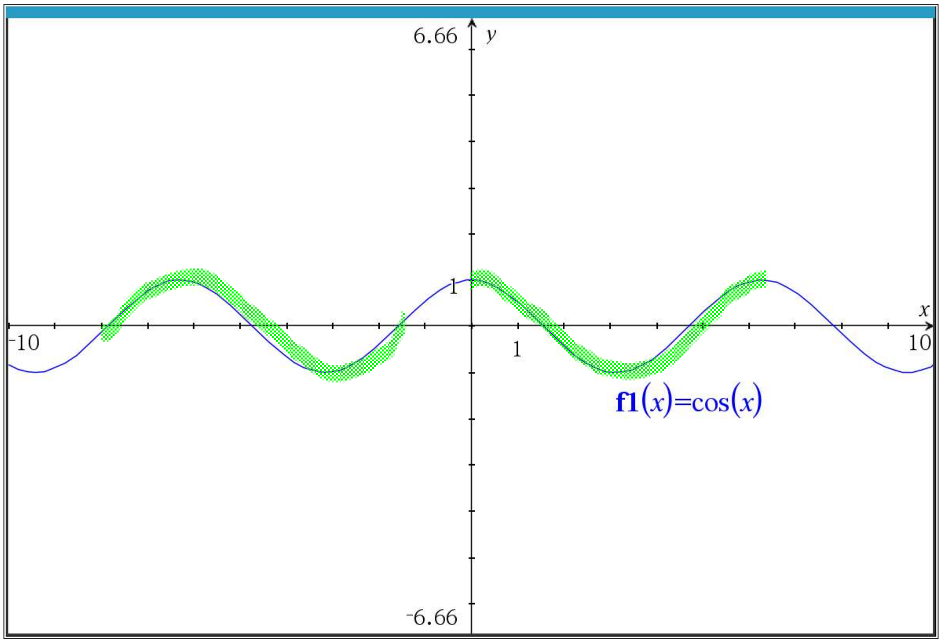
$t = T \Rightarrow u = 2wT + 2\phi$

$I = \frac{1}{2w}\int_{2\phi}^{2wT+2\phi} \cos u\, du$

$I = \frac{1}{2w} \sin(u)\Big]_{2\phi}^{2wT+2\phi}$

$I = \frac{1}{2w}\left[\sin(2wT+2\phi) - \sin(2\phi)\right]$

$\langle x^2(t) \rangle = \frac{A^2}{2} + \underbrace{\frac{A^2}{2wT}\left[\sin(2wT+2\phi) - \sin(2\phi)\right]}_{=0}$



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Let's work through this step by step:

(a) Average Power of $x(t) = A\cos(\omega t + \phi)$

We use the definition:

$$\langle x^2(t)\rangle = \frac{1}{T}\int_0^T x^2(t)\,dt$$

Substitute $x(t) = A\cos(\omega t + \phi)$:

$$x^2(t) = A^2\cos^2(\omega t + \phi)$$

Use the identity:

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$$

So:

$$x^2(t) = A^2 \cdot \frac{1+\cos(2\omega t + 2\phi)}{2}$$

Now compute the average over one period $T = \frac{2\pi}{\omega}$:

$$\langle x^2(t)\rangle = \frac{1}{T}\int_0^T \frac{A^2}{2}(1+\cos(2\omega t + 2\phi))\,dt$$

Split the integral:

$$\langle x^2(t)\rangle = \frac{A^2}{2T}\left[\int_0^T 1\,dt + \int_0^T \cos(2\omega t + 2\phi)\,dt\right]$$

The first integral is T, and the second is zero (since cosine over a full period averages to zero):

$$\langle x^2(t)\rangle = \frac{A^2}{2T}\cdot T = \frac{A^2}{2}$$

Average power is

(b) RMS Value as a Percentage of Amplitude

RMS is the square root of average power:

$$\text{RMS} = \sqrt{\frac{A^2}{2}} = \frac{A}{\sqrt{2}} \approx 0.707A$$

Expressed as a percentage of amplitude A:

$$\frac{A/\sqrt{2}}{A} \times 100\% = \frac{1}{\sqrt{2}} \times 100\% \approx 70.7\%$$

RMS is of the amplitude

Find $\lim_{x \to 2} (4x - 5)$

verify with ε-δ

$$\lim_{x \to 2} (4x - 5) = 4(2) - 5 = 8 - 5 = \boxed{3}$$

Let $\varepsilon > 0$

Find $\delta > 0$ such that

$$0 < |x - 2| < \delta \Rightarrow |(4x-5) - 3| < \varepsilon$$

$$|(4x-5) - 3| < \varepsilon$$

$$|(4x-5)-3| < \varepsilon$$

$$\iff |4x-5-3| < \varepsilon$$

$$\iff |4x-8| < \varepsilon$$

$$\iff 4|x-2| < \varepsilon$$

$$\iff |x-2| < \frac{\varepsilon}{4}$$

Let $\boxed{\delta = \frac{\varepsilon}{4}}$

$$\lim_{x\to\infty}\left(\frac{1}{x}+2\right)$$

Find limit and verify using formal definition

$$= \frac{1}{\infty}+2 = 0+2 = \boxed{2}$$

Let $\varepsilon > 0$

Find $M > 0$ such that

$$x > M \Rightarrow \left|\left(\frac{1}{x}+2\right)-2\right| < \varepsilon$$

$$\left|\left(\frac{1}{x}+2\right)-2\right| < \varepsilon$$

$$\iff \left|\frac{1}{x}+2-2\right| < \varepsilon$$

$$\iff \left|\frac{1}{}\right| < \varepsilon$$

$$\iff \left|\frac{1}{x}\right| < \varepsilon$$

$$\iff \frac{1}{|x|} < \varepsilon$$

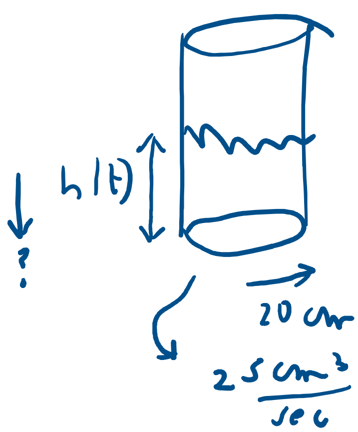
$$\iff \frac{1}{x} < \varepsilon \quad \text{(since } x \text{ gets large)}$$

$$\iff 1 < \varepsilon x$$

$$\iff \boxed{x > \frac{1}{\varepsilon}}$$

$\therefore$ Let $M = \frac{1}{\varepsilon}$

10. A cylindrical tank standing upright (with one circular base on the ground) has radius 20 cm. How fast does the water level in the tank drop when the water is being drained at 25 cm^3 /sec? Give exact answer.



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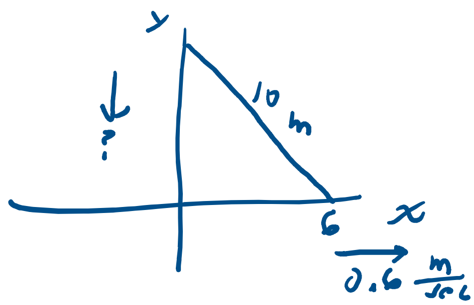
Please give me a related rates problem, without any hint or solution.

Sliding ladder related rates

A 10 m ladder leans against a vertical wall. The base is pulled away from the wall along the ground at a constant rate of 0.6 m/s. When the base is 6 m from the wall, find:

- **(i):** The rate at which the top of the ladder is sliding down the wall (in m/s).
- **(ii):** The rate of change of the angle θ between the ladder and the ground (in rad/s).



Let $x(t)$ = position (m) of ladder base at time t (sec)

Let $y(t)$ = height of ladder top (m) at time t (sec)

find $\frac{dy}{dt}$ when $x = 6$ m

$$\boxed{\frac{dx}{dt} = 0.6 \ \frac{m}{sec}}$$

$$x^2 + y^2 = 10^2$$

$$y^2 = 100 - x^2$$

$$\frac{d}{dt}(y^2) = \frac{d}{dt}(100 - x^2)$$

$$2y\frac{dy}{dt} = -2x\frac{dx}{dt}$$

$$\boxed{\frac{dy}{dt} = -\frac{2x}{2y}\frac{dx}{dt}}$$

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

$$\frac{dy}{dt} = -\frac{6}{y}(0.6)$$

$$y^2 = 100 - 6^2 = 100 - 36 = 64$$

$$\boxed{y = 8}$$

$$\frac{dy}{dt} = -\frac{6}{8}(0.6) \ \frac{m}{sec}$$

$$\frac{dy}{dt} = -\frac{3}{4}(0.6) \ \frac{m}{sec}$$

$(.3)$

$$\frac{dy}{dt} = -\frac{3}{4}(0.6)\ \frac{m}{sec}$$

$$\frac{dy}{dt} = -\frac{3(.3)}{2}$$

$$\frac{dy}{dt} = -\frac{.9}{2}\ \frac{m}{sec}$$

$$\boxed{\frac{dy}{dt} = -0.45\ \frac{m}{sec}}$$