Applications of Integrals

- 8.1 Area Between Curves page 266: 1, 5, 7, 9
- Average Value of a Function page 271: 1, 4, 7, 10

8.2

Memorize

The **average value** $\langle f \rangle$ of a function f over a closed interval [a,b] is:¹

$$\langle f \rangle = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \tag{8.3}$$

Туро

Find the average distance from the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ to the point (4,0).

Solution: Let d represent the distance from any point (x, y) on the ellipse to the point (4,0), as in Figure 8.2.3. If (x,y) is on the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ then $y^2 = 9(1 - \frac{x^2}{25}) = \frac{9}{25}(25 - x^2)$. So by the

epresent the distance from any point
$$(x, y)$$
 on point $(4, 0)$, as in Figure 8.2.3. If (x, y) is on $= 1$ then $y^2 = 9(1 - \frac{x^2}{25}) = \frac{9}{25}(25 - x^2)$. So by the d is given by
$$d^2 = (x - 4)^2 + (y - 0)^2 = (x - 4)^2 + y^2$$

$$= (x - 4)^2 + \frac{9}{25}(25 - x^2)$$
Figure 8.2.3 $\frac{x^2}{25} + \frac{y^2}{9} = 1$

$$= \frac{(x-4)^{2} + \frac{25}{25}(25-x^{2})}{25} = \frac{25(x-4)^{2} + 9(25-x^{2})^{2}}{25} = \frac{25x^{2} - 200x + 400 + 225 - 9x^{2}}{25}$$

$$= \frac{16x^{2} - 200x + 625}{25}$$

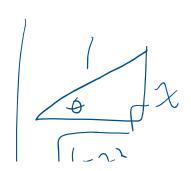
$$d^2 = \frac{(4x-25)^2}{25}$$
 , and so taking square roots gives

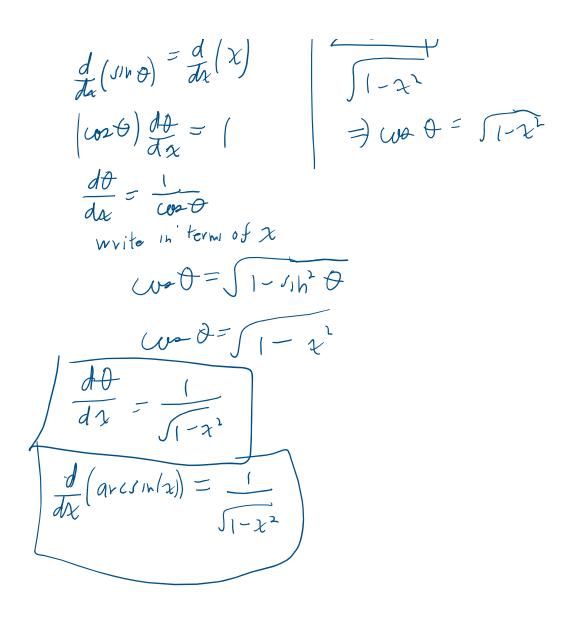
$$d = \pm \frac{4x - 25}{5} = -\frac{4x - 25}{5} = \frac{25 - 4x}{5}$$

derive
$$\frac{d}{dx} \left(\operatorname{arcsin}(x) \right)$$

Let $\frac{\partial}{\partial x} = \operatorname{arcsin}(x)$

Find $\frac{\partial}{\partial x} = \frac{\partial}{\partial x}(x)$





For Exercises 1-16, find the derivative of the given function y = f(x).

7.
$$y = \cot^{-1} 3x$$

$$\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

$$\frac{dy}{dx} = -\frac{1}{1 + (3x)^2}$$

$$\frac{dy}{dx} = -\frac{1}{1 + (3x)^2}$$

Here's a clean related rates problem for you:

Problem:

A balloon is rising vertically at a rate of 4 meters per second from a point on the ground that is 30 meters away from an observer. At the moment when the balloon is 50 meters above the ground, how fast is the distance between the balloon and the observer changing?

Let
$$L(t) = d_{i}xt$$
 ance i between at t ince t (see)

Let $Y(t) = d_{i}xt$ ance i between at t ince t (see)

Let $Y(t) = height of balloun at time t
 dt
 $dt$$

$$\frac{\partial L}{\partial t} = \frac{10}{200} \frac{\partial L}{\partial t}$$

$$\frac{\partial L}{\partial t} = \frac{(x(t))(4)}{L(t)}$$

$$\frac{\partial L}{\partial t$$

10*sqrt(34)/17=3.429971702850176

The distance is increasing at about 3.43 m/sec

Copilot

I get The distance is increasing at about 3.43 m/sec

Verification of the rate

Your answer is correct.

• Relationship: $s=\sqrt{30^2+y^2}$

Differentiate: $\frac{ds}{dt} = \frac{y}{\sqrt{30^2 + y^2}} \cdot \frac{dy}{dt}$

• Plug in: $y = 50, \ \frac{dy}{dt} = 4 \Rightarrow \frac{ds}{dt} = \frac{50 \cdot 4}{\sqrt{900 + 2500}} = \frac{200}{\sqrt{3400}} \approx 3.43 \text{ m/s}.$

Want a tougher one next—maybe with a changing angle (ladder against a wall) or a cone filling at a nonconstant rate?

Copilot did not define the variables explicitly, but the reasoning and calculations are correct.

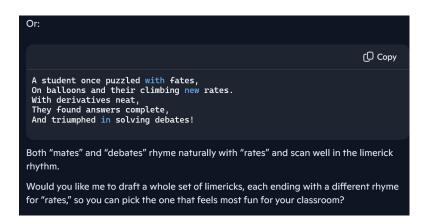
Copilot poetry

A student once puzzled with fates, On balloons and their climbing new rates. With derivatives neat, They found answers complete, And rejoiced at the joy of related rates! Yes, the rhyme and scan are good, but how is debates related to the topic?

You're right—"debates" doesn't really connect to related rates, even though it rhymes. Let's keep the rhyme but tie it back to the math theme. Here are a couple of stronger options:

C Copy

A student once puzzled with fates,
On balloons and their climbing new rates.
With derivatives neat,
They found answers complete,
And conquered their calculus plates!



Or

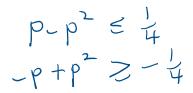
A student once puzzled with fates,
On balloons and their climbing new rates.
With derivatives neat,
They found answers complete,
And triumphed in solving their states!

Both "plates" and "states" rhyme naturally with "rates" and keep the limerick's rhythm,
while also nodding back to the context of tackling problems or "states of change."

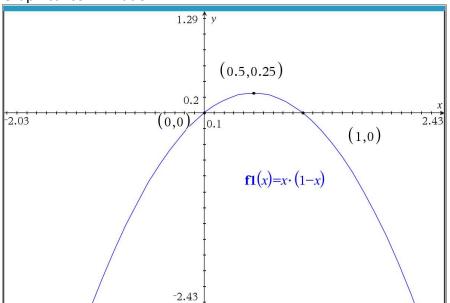
Would you like me to draft a whole mini-collection of math limericks, each ending
with a different rhyme for "rates," so you can pick the one that feels most fun and
classroom-ready?

4.1:2

2. Prove that for $0 \le p \le 1$, $p(1-p) \le \frac{1}{4}$.



Graphical confirmation.



Problem:

Evaluate the integral

$$\int \frac{x}{\sqrt{x^2 + 9}} \, dx$$

Let
$$I = \int \frac{x}{\sqrt{x^2 + 9}} dx$$

Then evaluate I = \(\frac{1}{5q^2+9} \) dx

$$U = x^2 + 9$$

$$= \int du = 2x dx = \int x dx = \frac{du}{x}$$

$$I_1 = 15 \frac{1}{4} du$$

$$|x| = (x^{2} + 9)^{2}$$

$$|x|$$

