

5.4 Integration by Substitution
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7 Analytic Geometry and Plain Curves

7.5 Hyperbolic Functions
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5.4: 3

A

For Exercises 1-24 evaluate the given integral.

$$3. \int \underbrace{(xe^{-x^2})}_{I_1} + \underbrace{x^2 \cos x^3}_{I_2} dx = I$$

$$I = I_1 + I_2$$

$$I_1 = \int x e^{-x^2} dx$$

$$u = -x^2 \\ \Rightarrow du = -2x dx \Rightarrow x dx = -\frac{du}{2}$$

$$I_1 = -\frac{1}{2} \int e^u du = \left(-\frac{1}{2}\right) e^u$$

$$\boxed{I_1 = -\frac{e^{-x^2}}{2}}$$

$$I_2 = \int x^2 \cos(x^3) dx$$

$$\text{Let } u = x^3 \\ \Rightarrow du = 3x^2 dx \Rightarrow x^2 dx = \frac{du}{3}$$

$$I_2 = \frac{1}{3} \int \cos u du$$

$$= \left(\frac{1}{3}\right) \sin u$$

$$\boxed{I_2 = \frac{\sin(x^3)}{3}}$$

$$\boxed{I = -\frac{e^{-x^2}}{2} + \frac{\sin(x^3)}{3} + C}$$

$$\boxed{I = -\frac{e^{-x^2}}{2} + \frac{\sin(x^3)}{3} + C}$$

5.4: 27

27. Recall from trigonometry that $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ for all x .

(a) Use the Fundamental Theorem of Calculus to evaluate $\int_0^\pi \sin^2 x \, dx$.

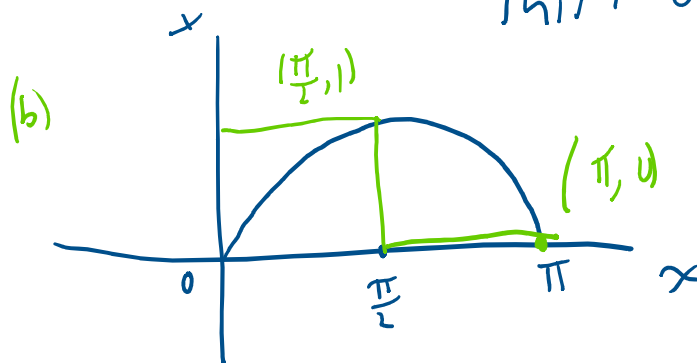
(b) Approximate the integral from part (a) by dividing the interval $[0, \pi]$ into $n = 2$ subintervals of equal length, $[0, \pi/2]$ and $[\pi/2, \pi]$, and finding the **exact** value of the sum of the areas of the rectangles whose heights are determined at the right endpoints of the subintervals.

(c) Repeat part (b) with $n = 3$.

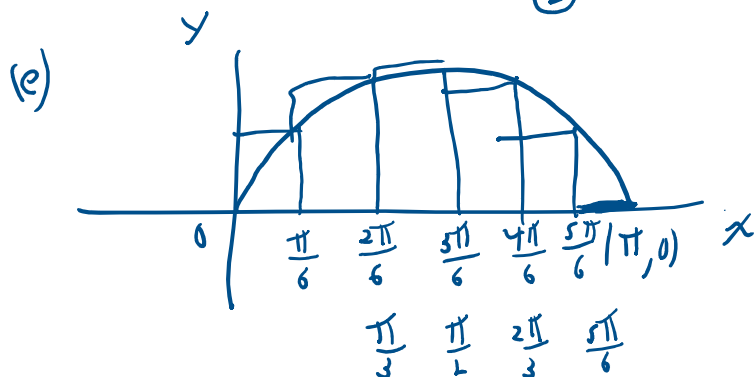
(d) Repeat part (b) with $n = 4$.

(e) Repeat part (b) with $n = 6$.

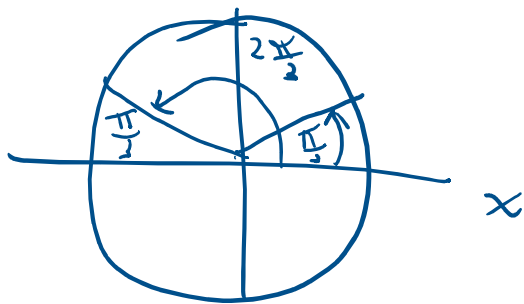
This is for $\int_0^\pi \sin x \, dx$



$$\begin{aligned} \int_0^\pi \sin(x) \, dx &\approx \left(\frac{\pi}{2}\right)(1) + \left(\frac{\pi}{2}\right)(0) \\ &\approx \left(\frac{\pi}{2}\right) \approx 1.57 \end{aligned}$$



$$\begin{aligned} \int_0^\pi \sin(x) \, dx &\approx \frac{\pi}{6} \left(\sin \frac{\pi}{3} + \sin \frac{2\pi}{3} + \sin \pi \right) \\ &= \frac{\pi}{6} \left(\frac{1}{2} + \frac{\sqrt{3}}{2} + 0 \right) \end{aligned}$$



$$\int_0^{\pi/2} \sin x \, dx \approx \frac{\pi}{6} (2 + \sqrt{3}) \approx 1.954$$

$$(\pi/6) * (2 + \sqrt{3}) = 1.954097233313706$$

Note: I forgot to square the sine function, but this is still a nice example.

7.5

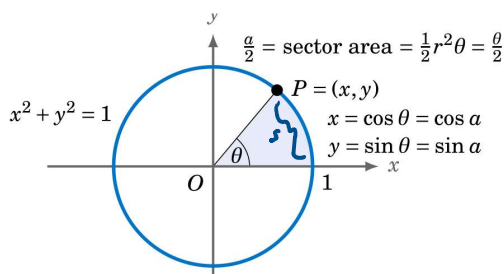


Figure 7.5.1 Circular

Arc length formula for a sector of a circle

$$s = r\theta$$

s = arc length
 r = radius of circle
 θ = central angle (in radians)

let A = area of sector

$$\frac{A}{\text{area of circle}} = \frac{A}{\pi r^2} = \frac{s}{2\pi r}$$

$$\Rightarrow A = \frac{\pi r^2 s}{2\pi r}$$

$$A = \frac{rs}{2}$$

$$A = \frac{r(r\theta)}{2}$$

$$\left[A = \frac{r^2\theta}{2} \right] = \frac{\theta}{2} \text{ for unit circle}$$

$$A = \frac{r^2 \theta}{2} = \frac{\theta}{2} \text{ for unit circle}$$

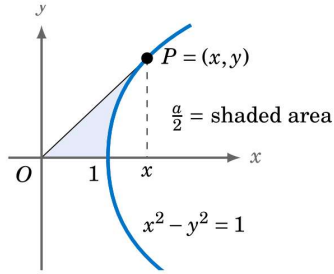


Figure 7.5.2 Hyperbolic

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The **hyperbolic sine**, **hyperbolic cosine**, **hyperbolic tangent**, **hyperbolic cotangent**, **hyperbolic secant** and **hyperbolic cosecant**, denoted by \sinh , \cosh , \tanh , \coth , sech and csch , respectively, are:

$$\begin{aligned} \sinh x &= \frac{e^x - e^{-x}}{2} \quad \text{for all } x & \cosh x &= \frac{e^x + e^{-x}}{2} \quad \text{for all } x \\ \tanh x &= \frac{\sinh x}{\cosh x} \quad \text{for all } x & \coth x &= \frac{1}{\tanh x} \quad \text{for all } x \neq 0 \\ \operatorname{sech} x &= \frac{1}{\cosh x} \quad \text{for all } x & \operatorname{csch} x &= \frac{1}{\sinh x} \quad \text{for all } x \neq 0 \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}(\sinh(x)) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x}{2} + \frac{e^{-x}}{2} = \cosh(x) \\ \frac{d}{dx}(\cosh(x)) &= \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x}{2} - \frac{e^{-x}}{2} = \sinh(x) \end{aligned}$$

Supplied or be able to verify

$$\begin{aligned} \cosh^2 x - \sinh^2 x &= 1 & \tanh^2 x + \operatorname{sech}^2 x &= 1 & \coth^2 x - \operatorname{csch}^2 x &= 1 \\ \sinh(-x) &= -\sinh x & \cosh(-x) &= \cosh x & \tanh(-x) &= -\tanh x \\ \operatorname{sech}(-x) &= \operatorname{sech} x & \operatorname{csch}(-x) &= -\operatorname{csch} x & \coth(-x) &= -\coth x \\ \sinh(u \pm v) &= \sinh u \cosh v \pm \cosh u \sinh v & \sinh 2x &= 2 \sinh x \cosh x \\ \cosh(u \pm v) &= \cosh u \cosh v \pm \sinh u \sinh v & \cosh 2x &= \cosh^2 x + \sinh^2 x \\ \tanh(u \pm v) &= \frac{\tanh u \pm \tanh v}{1 \pm \tanh u \tanh v} & \tanh 2x &= \frac{2 \tanh x}{1 + \tanh^2 x} \end{aligned}$$

Supplied or be able to verify.

$$\begin{aligned} \frac{d}{dx}(\sinh x) &= \cosh x & \frac{d}{dx}(\operatorname{csch} x) &= -\operatorname{csch} x \coth x \\ \frac{d}{dx}(\cosh x) &= \sinh x & \frac{d}{dx}(\operatorname{sech} x) &= -\operatorname{sech} x \tanh x \end{aligned}$$

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

$$\int \cosh x \, dx = \sinh x + C$$

$$\int \sinh x \, dx = \cosh x + C$$

$$\int \operatorname{sech}^2 x \, dx = \tanh x + C$$

$$\int \operatorname{csch} x \coth x \, dx = -\operatorname{csch} x + C$$

$$\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + C$$

$$\int \operatorname{csch}^2 x \, dx = -\coth x + C$$

Supplied

function	$\sinh^{-1} x$	$\cosh^{-1} x$	$\tanh^{-1} x$	$\operatorname{csch}^{-1} x$	$\operatorname{sech}^{-1} x$	$\coth^{-1} x$
domain	all x	$x \geq 1$	$ x < 1$	all $x \neq 0$	$0 < x \leq 1$	$ x > 1$
range	all y	$y \geq 0$	all y	all $y \neq 0$	$y \geq 0$	all $y \neq 0$

Supplied

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$$

$$\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x} \quad \text{for } |x| < 1$$

$$\operatorname{sech}^{-1} x = \ln \frac{1 + \sqrt{1 - x^2}}{x} \quad \text{for } 0 < x \leq 1$$

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \quad \text{for } x \geq 1$$

$$\coth^{-1} x = \frac{1}{2} \ln \frac{x+1}{x-1} \quad \text{for } |x| > 1$$

$$\operatorname{csch}^{-1} x = \ln \left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1} \right) \quad \text{for } x \neq 0$$

Supplied or verify, given appropriate formulas

Your Name MTH 263 quiz 9 Write each problem

1. Evaluate analytically, using u-substitution. $I = \int_2^3 x^2(x^3 + 1)dx$

$$\begin{aligned} u &= x^3 + 1 \\ \Rightarrow du &= 3x^2 dx \Rightarrow x^2 dx = \frac{du}{3} \\ x=2 &\Rightarrow u = 2^3 + 1 = 9 \\ x=3 &\Rightarrow u = 3^3 + 1 = 28 \\ I &= \frac{1}{3} \int_9^{28} u \, du \\ &= \left(\frac{1}{3} \right) \left(\frac{u^2}{2} \right)_9^{28} \end{aligned}$$

$$\therefore \quad \therefore \quad 784 - 81$$

Alternative

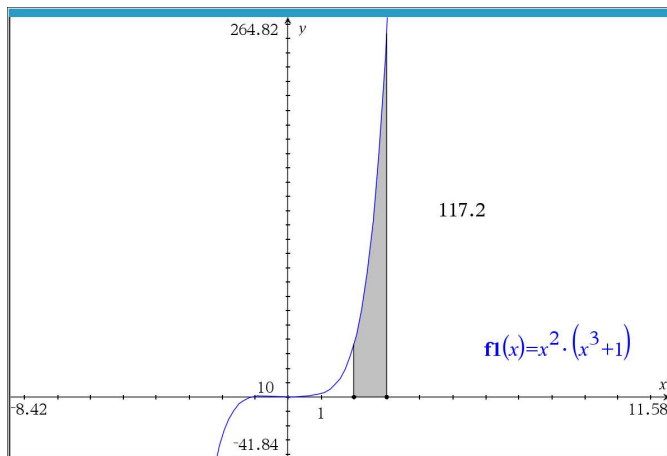
$$\begin{aligned} I &= \frac{1}{3} \int_2^3 u \, du \\ &= \left(\frac{1}{3} \right) \left(\frac{u^2}{2} \right) \Big|_{x=2}^3 \\ &= \left(\frac{1}{6} \right) (x^3 + 1)^2 \Big|_2^3 \\ &= \left(\frac{1}{6} \right) (28^2 - 9^2) \\ &= \left(\frac{1}{6} \right) 703 = \boxed{\frac{703}{6}} \end{aligned}$$

$$\boxed{703} \quad \therefore \quad \therefore$$

$$= \left(\frac{1}{6}\right)(28^2 - 9^2) = \frac{784 - 81}{6} = \frac{703}{6} \approx 117.2$$

703/6=117.1667 28^2-81=703

2. Evaluate #1 / graphically. Sketch a labeled graph.

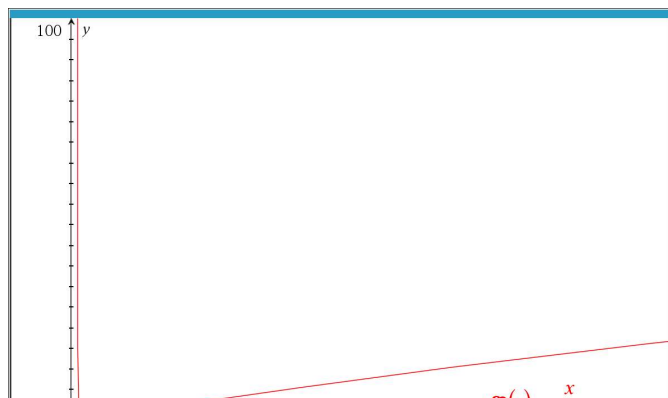


3. Find $\lim_{x \rightarrow \infty} \left(\frac{x}{\ln(x)} \right)$

$$\stackrel{[\frac{\infty}{\infty})}{=} \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(x)}{\frac{d}{dx}(\ln x)} = \lim_{x \rightarrow \infty} \frac{1}{\frac{1}{x}}$$

$$= \lim_{x \rightarrow \infty} x = \infty \text{ (dne)}$$

4. Graph $y = \left(\frac{x}{\ln(x)} \right)$ and compare the graphical limit with your answer to #3. Sketch a labeled graph.

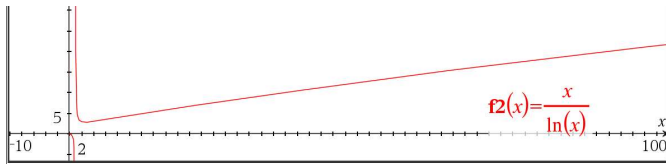


$$y' = \frac{d}{dx} \left(\frac{x}{\ln(x)} \right)$$

$$= \frac{(\ln(x)) \frac{dx}{dx} - x \frac{d}{dx}(\ln x)}{(\ln(x))^2}$$

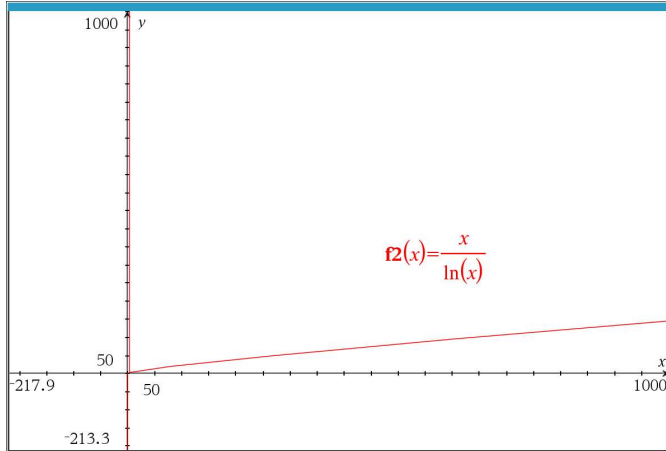
$$= \ln(x) - x \left(\frac{1}{x} \right)$$

$$= \ln(x) - 1 > 0 \text{ for } x > 1$$



$$= \ln(x) - 1 > 0 \\ \text{for } x > 1$$

$$\therefore \frac{x}{\ln x} \text{ increases on } (1, \infty)$$



This is nice, but it does not prove that the function goes to infinity.

It appears that no matter how much we zoom out, $f(x)$ approaches infinity.

After class notes

$$I = \int_0^{\pi} x \cos(x^2) dx$$

$$\text{Let } u = x^2$$

$$\Rightarrow du = 2x dx \Rightarrow x dx = \frac{du}{2}$$

$$x = 0 \Rightarrow u = 0$$

$$x = \pi \Rightarrow u = \pi^2$$

$$I = \frac{1}{2} \int_0^{\pi^2} \cos(u) du$$

$$= \frac{1}{2} \left[\sin(u) \right]_0^{\pi^2}$$

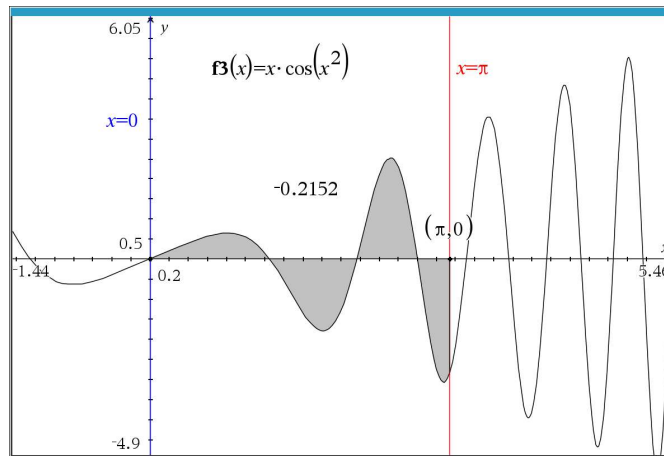
$$= \frac{1}{2} \left(\sin(\pi^2) - \sin(0) \right)$$

$$= \frac{\sin(\pi^2)}{2} - \frac{0}{2}$$

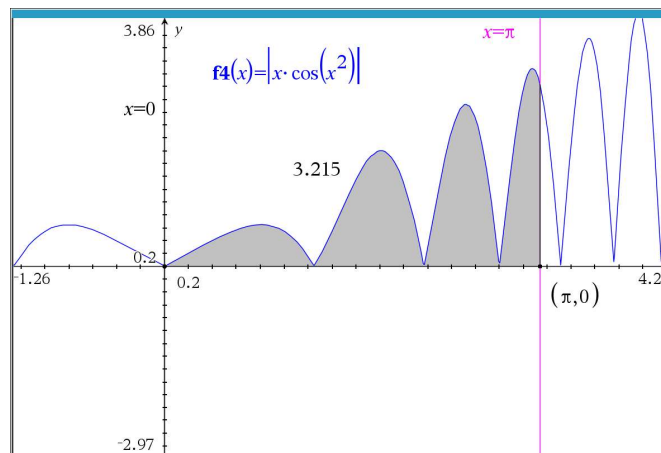
$$I = \frac{\sin(\pi^2)}{2} \approx -0.215$$

$$\frac{\sin(\pi^2)}{2} \rightarrow \text{Decimal}$$

-0.215151



Here is the total shaded area.



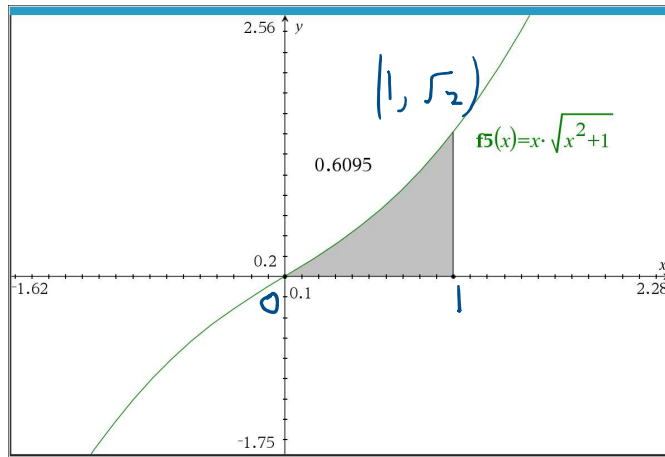
Copilot

Please give me an example of a definite integral requiring u-substitution. Let me try it before you do the integration.

Great! Here's a definite integral that calls for a classic u -substitution:

$$\int_0^1 x \cdot \sqrt{1+x^2} \, dx$$

This one is perfect for substitution because the derivative of $1+x^2$ appears as part of the integrand. Give it a shot—try choosing a substitution for u , rewriting the bounds, and transforming the integral. Let me know when you're ready to compare notes!



Area of $\Delta = \left(\frac{1}{2}\right)(1)(\sqrt{2}) = \frac{\sqrt{2}}{2} \approx 0.71$
 right overestimate

$\text{Sqrt}(2)/2 = 0.707106781186548$

$$I = \int_0^1 x \sqrt{x^2 + 1} dx$$

$$u = x^2 + 1$$

$$\Rightarrow du = 2x dx \Rightarrow x dx = \frac{du}{2}$$

$$x=0 \Rightarrow u=1$$

$$x=1 \Rightarrow u=2$$

$$I = \frac{1}{2} \int_1^2 u^{\frac{1}{2}} du$$

$$= \frac{1}{2} \left[\frac{2u^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^2$$

$$= \left(\frac{1}{3}\right) \left(2^{3/2} - 1 \right)$$

$$(1/3) * (2^{3/2} - 1) = 0.6095$$