4.4 The Mean Value Theorem
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5 The Integral
5.1 The Indefinite Integral
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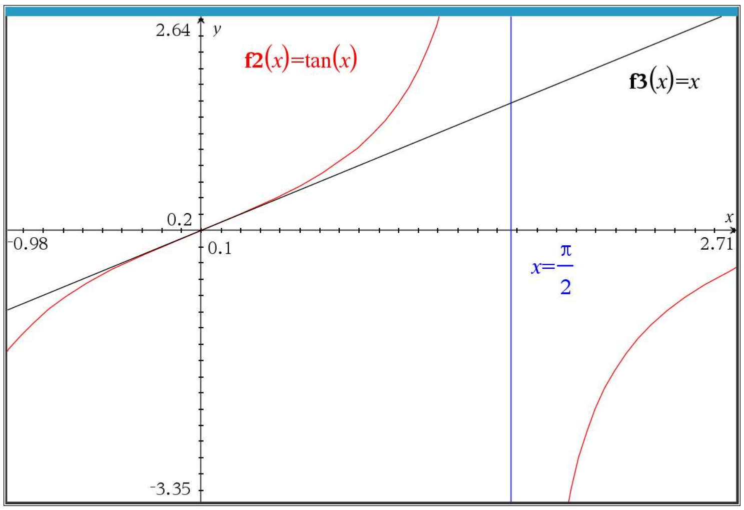
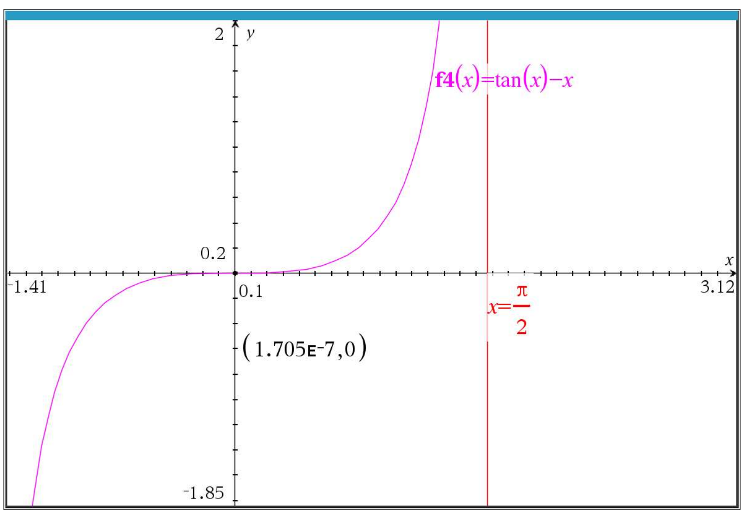
4.4: 5

5. Show that $\tan x \geq x$ for all $0 \leq x < \frac{\pi}{2}$.





Let $f(x) = (\tan x) - x$

$f'(x) = \sec^2(x) - 1$

$0 \leq x < \frac{\pi}{2}$

$f(0) = \tan 0 - 0$
$= 0$

$= \frac{1}{\cos^2 x} - 1 > 0$ on $(0, \frac{\pi}{2})$

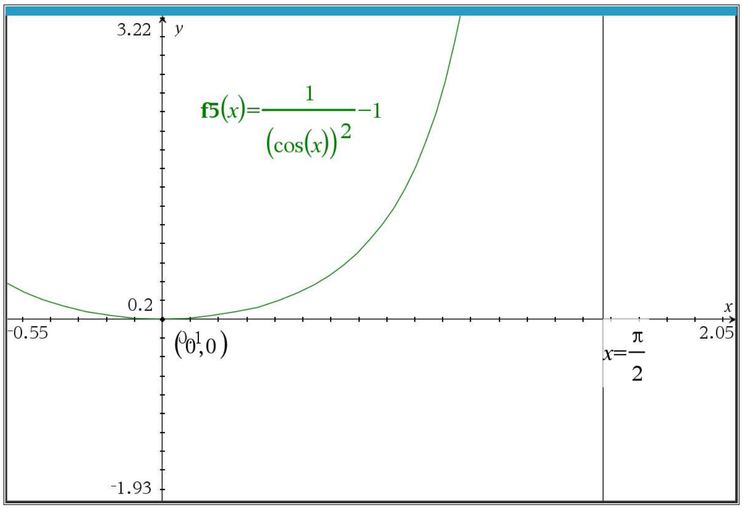
$0 < \cos^2 x < 1$

, $f(x)$ is increasing on $(0, \frac{\pi}{2})$

$0 < \cos^2 x < 1$

$\therefore f(x)$ is increasing on $\left(0, \frac{\pi}{2}\right)$

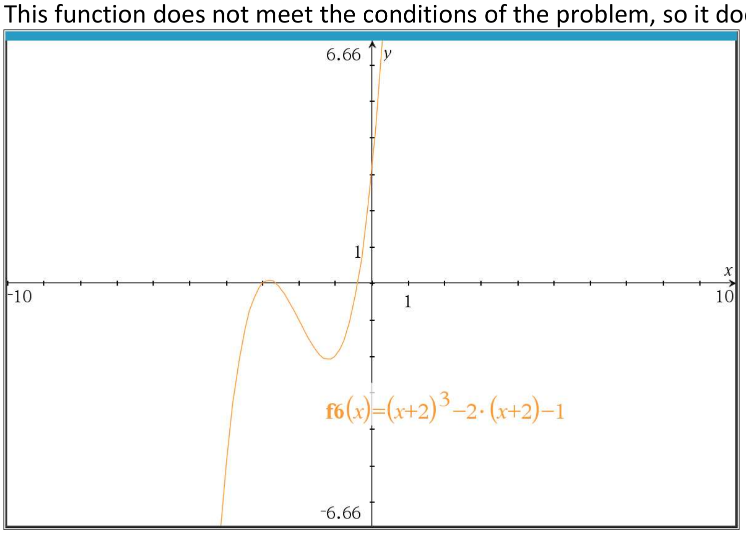
$\therefore f(x) \geq 0$ on $\left[0, \frac{\pi}{2}\right)$



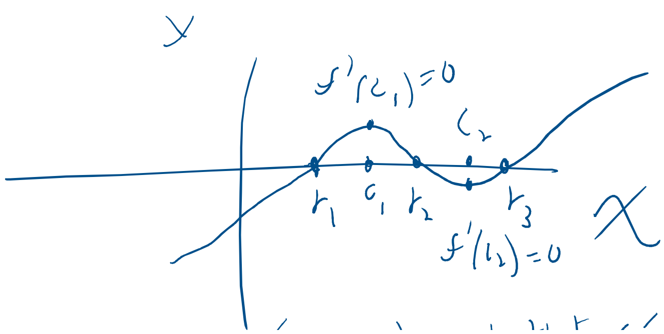
4.4: 7

7. Use Rolle's Theorem to show that for all constants a and b with $a > 0$, $f(x) = x^3 - ax + b$ can not have three positive roots. Also, show that it can not have three negative roots.

This function does not meet the conditions of the problem, so it does not contradict the proposition.



Assume the opposite of what we want.
That is, assume there exist 3 positive roots



$f(x) = x^3 - ax + b$

$f'(x) = ?$

$f(r_2)=0$

$\exists c_1 \in (r_1, r_2)$ such that $f'(c_1)=0$

$\exists c_2 \in (r_2, r_3)$ such that $f'(c_2)=0$

Think about this

Look for a contradictions

5.1

Memorize

$$\begin{tikzcd} s(t) \arrow[r, bend left, "\frac{d}{dt}"] & v(t) \arrow[l, bend left, "?"] \arrow[r, bend left, "\frac{d}{dt}"] & a(t) \arrow[l, bend left, "?"] \end{tikzcd}$$

Figure 5.1.1 Differentiation vs antidifferentiation for motion functions

$s(t)$ = position at time t

$v(t)$ = velocity at time $t = \dfrac{ds}{dt}$

$a(t)$ = acceleration at time $t = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2}$

Memorize

An **antiderivative** $F(x)$ of a function $f(x)$ is a function whose derivative is $f(x)$. In other words, $F'(x) = f(x)$.

$$\frac{d}{dx}(3x+2) = 3$$

$$\frac{d}{dx}(3x) = 3$$

Memorize

Suppose that $F(x)$ and $G(x)$ are antiderivatives of a function $f(x)$. Then $F(x)$ and $G(x)$ differ only by a constant. That is, $F(x) = G(x) + C$ for some constant C.

Memorize

To find *all* antiderivatives of a function, it is necessary only to find *one* antiderivative and then add a generic constant to it.

Memorize

The **indefinite integral** of a function $f(x)$ is denoted by

$$\int f(x)\,dx$$

and represents the entire family of antiderivatives of $f(x)$.

Memorize

$$\frac{d}{dx}\left(\int f(x)\,dx\right) = f(x)$$

integrand

You might be wondering what the integral sign in the indefinite integral represents, and why an infinitesimal dx is included. It has to do with what an infinitesimal represents: an infinitesimal "piece" of a quantity. For an antiderivative $F(x)$ of a function $f(x)$, the infinitesimal (or differential) dF is given by $dF = F'(x)\,dx = f(x)\,dx$, and so

$$F(x) = \int f(x)\,dx = \int dF\,.$$

The integral sign thus acts as a summation symbol: it sums up the infinitesimal "pieces" dF of the function $F(x)$ at each x so that they add up to the entire function $F(x)$. Think of it as similar to the usual summation symbol Σ used for *discrete* sums; the integral sign $\int$ takes the sum of a *continuum* of infinitesimal quantities instead.

Finding (or **evaluating**) the indefinite integral of a function is called **integrating** the function, and **integration** is antidifferentiation.

Memorize

Power Formula: $\displaystyle \int x^n\,dx = \begin{cases} \dfrac{x^{n+1}}{n+1} & \text{if } n \neq -1 \\ \ln|x| & \text{if } n = -1 \end{cases}$

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \left(\frac{1}{n+1}\right)\frac{d}{dx}\left(x^{n+1}\right)$$

$$= \left(\frac{1}{\cancel{n+1}}\right)(\cancel{n+1})\left(x^{n}\right)$$

$$= x^n$$

Memorize

Let f and g be functions and let k be a constant. Then:

1. $\int k\,f(x)\,dx = k\int f(x)\,dx$

2. $\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$

3. $\int (f(x)-g(x))\,dx = \int f(x)\,dx - \int g(x)\,dx$

Memorize

For any functions $f_1, \ldots, f_n$ and constants $k_1, \ldots, k_n$,

$$\int (k_1 f_1(x) + \cdots + k_n f_n(x))\,dx = k_1 \int f_1(x)\,dx + \cdots + k_n \int f_n(x)\,dx\,. \tag{5.1}$$

Memorize

$$\int \cos x\,dx = \sin x + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \csc x \cot x\,dx = -\csc x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

Memorize

$$\int e^x\,dx = e^x + C$$

Be able to derive

Free fall motion: At time $t \geq 0$:

$$\begin{aligned}
\text{acceleration:} \quad a(t) &= -g \\
\text{velocity:} \quad v(t) &= -gt + v_0 \\
\text{position:} \quad s(t) &= -\tfrac{1}{2}gt^2 + v_0 t + s_0 \\
\text{initial conditions:} \quad s_0 &= s(0),\ v_0 = v(0)
\end{aligned}$$

Evaluate $\int (2x^3 - \pi x + 3)\,dx$

Check by differentiation.

$$= \int 2x^3\,dx - \int \pi x\,dx + \int 3\,dx \quad \text{[sum/difference rule]}$$

$$= 2\int x^3\,dx - \pi\int x\,dx + 3\int dx \quad \text{[constant multiple rule]}$$

$$= 2\left(\frac{x^4}{4}\right) - \pi\frac{x^2}{2} + 3x + C$$

$$= \boxed{\frac{x^4}{2} - \frac{\pi x^2}{2} + 3x + C}$$

Preview of u-substitution

$$I = \int \cos(3x)\,dx$$

$$\int \cos(x)\,dx = \sin x$$

$$I \stackrel{?}{=} \sin(3x) + C$$

$$\frac{d}{dx}(\sin(3x)) = \cos(3x)\frac{d}{dx}(3x)$$

$$= 3\cos(3x)$$

$$\Rightarrow I = \frac{\sin(3x)}{3} + C$$

Check $\frac{d}{dx}\left(\frac{\sin(3x)}{3} + C\right)$

$$= \frac{1}{3}\frac{d}{dx}(\sin 3x) + 0$$

$$= \frac{1}{3}(\cos 3x)\left(\frac{d}{dx}(3x)\right)$$

$$= \frac{1}{3}(\cos 3x)(3)$$

$= \cos 3x$

Your name MTH 263 quiz 7

1. Evaluate $\int e^x\,dx$.

$\boxed{e^x + c}$

check $\frac{d}{dx}(e^x + c) = \frac{d}{dx}(e^x) + \frac{d}{dx}(c)$

$= e^x + 0$

$= e^x$

2. Evaluate $\int(\sqrt{x} - 5)\,dx$.

Let $I = \int (\sqrt{x} - 5)\,dx$

$I = \int (x^{\frac{1}{2}} - 5)\,dx = \int x^{\frac{1}{2}}\,dx - \int 5\,dx$

$= \frac{2x^{\frac{3}{2}}}{3} - 5x + c$

3. Find the limit $\lim\limits_{x\to\infty}\left(\frac{x^2}{\ln(x)}\right)$.

$\lim\limits_{x\to\infty} \frac{x^2}{\ln(x)} \left[\frac{\infty}{\infty}\right] \overset{(L)}{=} \lim\limits_{x\to\infty} \frac{\frac{d}{dx}(x^2)}{\frac{d}{dx}(\ln(x))}$

$= \lim\limits_{x\to\infty} \frac{2x}{\frac{1}{x}} = \lim\limits_{x\to\infty} 2x^2 = \boxed{\infty\ (\text{dne})}$