10-29-25 MTH 263

4.2 Curve Sketching
page 118: 1, 3, 9

4.4 The Mean Value Theorem
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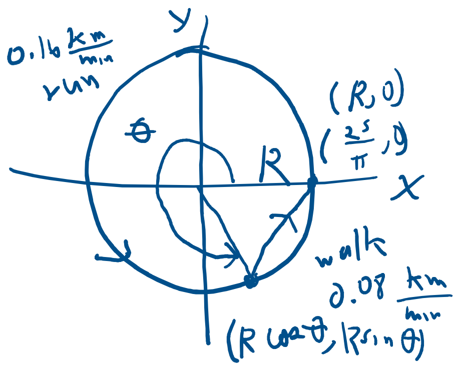
17. A certain jogger can run 0.16 km/min, and walks at half that speed. If he runs along a circular trail with circumference 50 km and then—before completing one full circle—walks back straight across to his starting point, what is the maximum time he can spend on the run/walk?

$d = rt \Rightarrow t = \frac{d}{r}$

distance = (rate) (time)



Let R = radius

$2\pi R = 50\text{ km}$

$R = \frac{50\text{ km}}{2\pi}$

$R = \frac{25\text{ km}}{\pi}$

arc length $s = R\theta = \left(\frac{25}{\pi}\text{ km}\right)\theta$

$$\boxed{t_{run} = \frac{s}{r_{run}} = \frac{R\theta}{r_{run}}}$$

Let t_{run} = run time

t_{walk} = walk time

$t = t_{run} + t_{walk}$

maximize t

Copilot

1. Plain Text Version (no symbols, just words and numbers)

- Running speed = 0.16 km per minute
- Walking speed = 0.08 km per minute
- Circle circumference = 50 km, so radius = 25/π km

Suppose the jogger runs a distance **s** km along the circle.
The central angle is (2πs)/50.
The straight-line distance back to the start (the chord) is:
d = (50/π) × sin(πs/50).

Total time = (running time) + (walking time)
= (s ÷ 0.16) + (d ÷ 0.08).

This simplifies to:
T(s) = 6.25s + (625/π) × sin(πs/50).

Differentiate:
T′(s) = 6.25 + 12.5 × cos(πs/50).

Set T′(s) = 0 → cos(πs/50) = −1/2.
So πs/50 = 2π/3 or 4π/3.
That gives s = 100/3 ≈ 33.3 km or s = 200/3 ≈ 66.7 km.
But s must be less than 50, so only s ≈ 33.3 km is valid.

At s = 33.3 km, chord length d ≈ 13.78 km.
Total time = (33.3 ÷ 0.16) + (13.78 ÷ 0.08) ≈ 380.6 minutes.

Maximum time ≈ 381 minutes (about 6 hours 21 minutes).

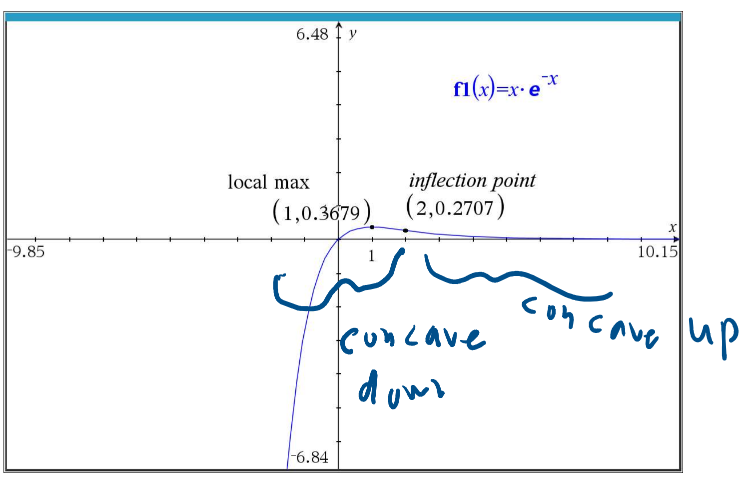
textbook answer

17. 380.62 minutes

4.2: 3

For Exercises 1-8 sketch the graph of the given function. Find all local maxima and minima, inflection points, where the function is increasing or decreasing, where the function is concave up or concave down, and indicate any asymptotes.

3. $f(x) = xe^{-x}$



$$f'(x) = \frac{d}{dx}\left(xe^{-x}\right)$$

$$= x(-e^{-x}) + e^{-x}$$

$$\boxed{f'(x) = -xe^{-x} + e^{-x}}$$

$$\boxed{f'(x) = -xe^{-x} + e^{-x}}$$

$$-xe^{-x} + e^{-x} = 0$$

$$e^{-x}(-x+1) = 0$$

$$e^{-x} \neq 0$$

$$\therefore -x+1 = 0$$

$$\boxed{x = 1}$$

$$f''(x) = -e^{-x} \cdot x(-e^{-x}) - e^{-x}$$

$$= -e^{-x} + xe^{-x} - e^{-x}$$

$$= -2e^{-x} + xe^{-x}$$

$$\boxed{f''(x) = e^{-x}(-2+x)}$$

$$f''(1) = e^{-1}(-2+1)$$

$$= e^{-1}(-1) < 0$$

$\therefore$ local max at $\boxed{x = 1}$

$$f(1) = 1 \cdot e^{-1} = \boxed{e^{-1} = \frac{1}{e}}$$

$\left(e^{-1}\right)$ ▸Decimal 0.367879

$$e^{-x}(-2+x) = 0$$

$$\boxed{x = 2}$$

Concave down $(-\infty, 2)$

Concave up $(2, \infty)$

increasing $(-\infty, 1)$

increasing $(-\infty, 1)$

decreasing $(1, \infty)$

Quiz 6

① $\lim_{x \to \infty} x e^{-x} = ?$

$(\infty)(0)$

$$= \lim_{x \to \infty} \frac{x}{e^x} \underset{(L)}{=} \lim_{x \to \infty} \frac{\frac{d}{dx}(x)}{\frac{d}{dx}(e^x)}$$

$\left(\frac{\infty}{\infty}\right)$

$$= \lim_{x \to \infty} \frac{1}{e^x} = \frac{1}{\infty} = \boxed{0}$$

② $\lim_{x \to 0} x e^{-x} = ?$

$\left[\frac{0}{1}\right]$ not indeterminate

$$= \lim_{x \to 0} \frac{x}{e^x} \underset{(L)}{=} \lim_{x \to 0} \frac{\frac{d}{dx}(x)}{\frac{d}{dx}(e^x)}$$

$$= \lim_{x \to 0} \frac{1}{e^x} = \frac{1}{e^0} = \frac{1}{1} = 1$$

$$\lim_{x \to 0} x e^{-x} = 0 \cdot e^{-0} = 0 \cdot 1 = \boxed{0}$$

4.2: 9

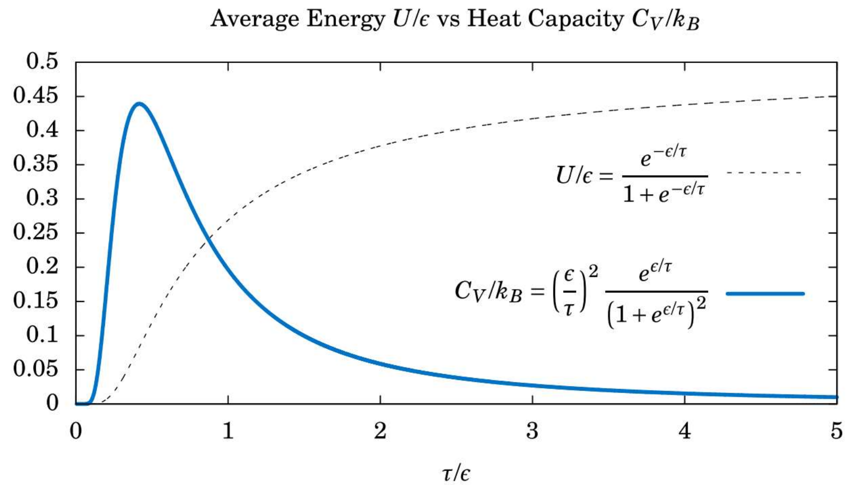
9. Write U/ϵ and C_V/k_B from Example 4.13 as functions of $x = \tau/\epsilon$. You do not need to sketch the graphs.

Example 4.13

For a single particle with two states—energy 0 and energy ϵ—in thermal contact with a reservoir at temperature τ, the average energy U and heat capacity C_V are given by

$$U = \epsilon \frac{e^{-\epsilon/\tau}}{1+e^{-\epsilon/\tau}} \quad \text{and} \quad C_V = k_B \left(\frac{\epsilon}{\tau}\right)^2 \frac{e^{\epsilon/\tau}}{\left(1+e^{\epsilon/\tau}\right)^2}$$

where $k_B \approx 1.38065 \times 10^{-23}$ J/K is the *Boltzmann constant*. The graph below shows both quantities as functions of τ/ϵ (*not* ϵ/τ, as you might expect). See Exercise 9.



$$U = \epsilon \frac{e^{-\epsilon/\tau}}{1+e^{-\epsilon/\tau}} \quad \text{and} \quad C_V = k_B \left(\frac{\epsilon}{\tau}\right)^2 \frac{e^{\epsilon/\tau}}{\left(1+e^{\epsilon/\tau}\right)^2}$$

Let $x = \frac{\tau}{\varepsilon}$

$\Rightarrow \frac{\varepsilon}{\tau} = \frac{1}{x}$

$$\frac{U}{\varepsilon} = \frac{\varepsilon e^{-\frac{1}{x}}}{\varepsilon(1+e^{-\frac{1}{x}})}$$

$$\boxed{\frac{U}{\varepsilon} = \frac{e^{-\frac{1}{x}}}{1+e^{-\frac{1}{x}}}}$$

$$\frac{C_v}{k_b} = \frac{k_B\left(\frac{1}{x}\right)^2}{k_B} \frac{e^{\frac{1}{x}}}{\left(1+e^{\frac{1}{x}}\right)^2}$$

$$\frac{C_v}{k_b} = \frac{\left(\frac{1}{x}\right)^2 e^{-\frac{1}{x}}}{(1+e^{1/x})^2}$$

$$\boxed{\frac{C_v}{k_b} = \frac{e^{-\frac{1}{x}}}{x^2(1+e^{1/x})^2}}$$

4.3
supplied

Mean Value Theorem: Let a and b be real numbers such that $a < b$, and suppose that f is a function such that
(a) f is continuous on $[a, b]$, and

(b) f is differentiable on (a, b).

Then there is at least one number c in the interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \qquad (4.3)$$

Considerate value theorem?
Above average value theorem?

Course Content Summary

Applications of Differentiation

- Calculate local and absolute maximum and minimum values of a function
- Apply Rolle's Theorem and Mean Value Theorem to study properties of a function
- Find critical points, and intervals of increasing and decreasing values of a function
- Find points of inflection and intervals of different concavities
- Sketch a curve for a given function
- Apply rules of differentiation to solve optimization problems
- Find antiderivatives for basic functions using knowledge of derivatives

Figure 4.4.1 below shows the geometric interpretation of the theorem:

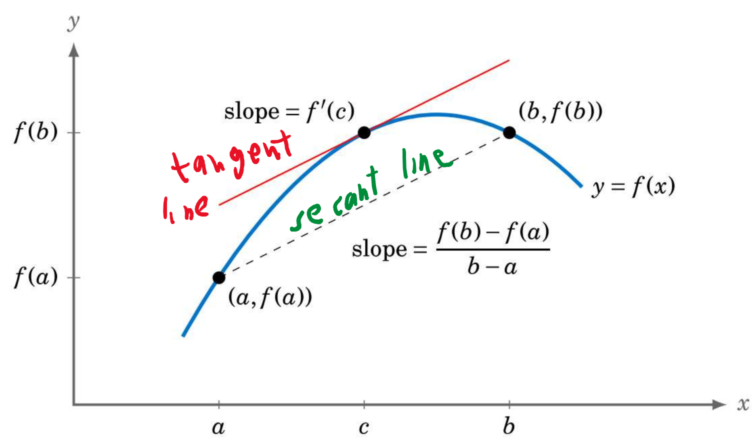


Figure 4.4.1 Mean Value Theorem: parallel tangent line and secant line

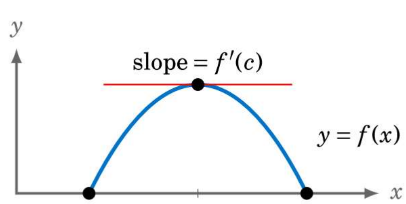
Supplied

Rolle's Theorem: Let a and b be real numbers such that $a < b$, and suppose that f is a function such that
(a) f is continuous on $[a, b]$,

(b) f is differentiable on (a, b), and

(c) $f(a) = f(b) = 0$.

Then there is at least one number c in the interval (a, b) such that $f'(c) = 0$.



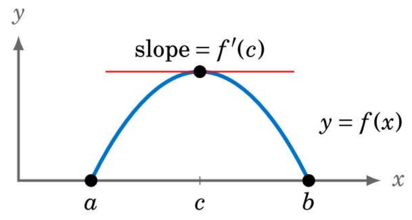
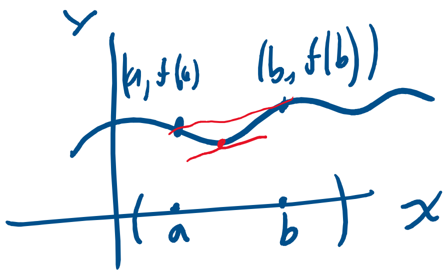


Figure 4.4.2

Memorize

> If f is a differentiable function on an interval I such that $f'(x) = 0$ for all x in I, then f is a constant function on I.



$y = f(x)$

$f'(x) = 0$ on I

I

contrapositive

$$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$$

implies   equivalent   not

Assume $f(x)$ not constant

Try to deduce a contradiction

$\Rightarrow \exists\ a, b \in I$ such that

$f(a) \neq f(b)$

MVT $\Rightarrow \exists\ c \in (a, b)$

such that $f'(c) = \dfrac{f(b) - f(a)}{b - a} \neq 0$

but $f'(x) = 0\ \forall x \in I$

Contradiction

$\therefore f(x)$ is constant on I

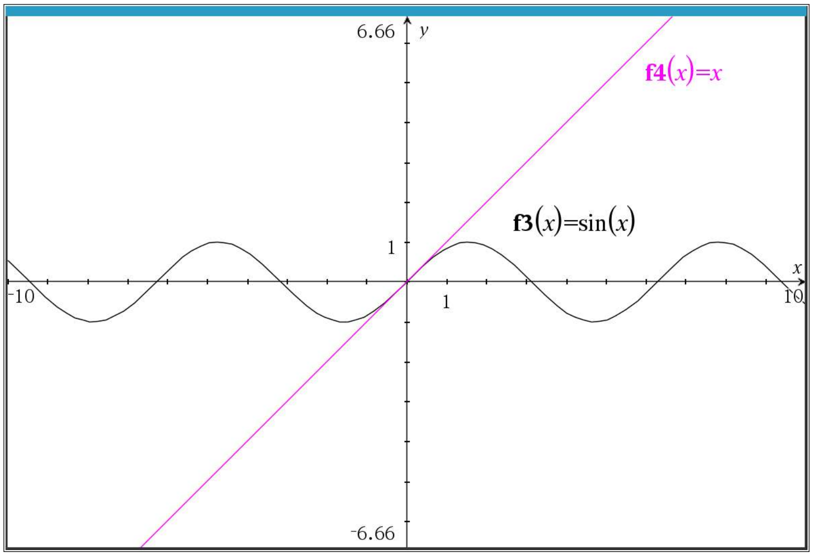
QED

Memorize

Let f be a differentiable function on an interval I. Then:

(a) If $f' > 0$ on I then f is increasing on I.

(b) If $f' < 0$ on I then f is decreasing on I.



Example 4.16

Show that $\sin x \leq x$ for all $x \geq 0$.

Use MVT