10-27-25 MTH 263

4 Applications of Derivatives

4.1 Optimization

page 108: 1, 3, 7, 13, 17

4.2 Curve Sketching

page 118: 1, 3, 9

Exam 2		stem & leaf		
50	mean			A-0
46	median	8	7	B-1
16.24808	st. dev	7	0	C-1
21	min	6	268	D-3
87	max	5	07	F- 12
17	count	4	0014569	
		3	9	
		2	15	

Exam 1		stem & leaf		
50.21053	mean			A-0
53	median	8	6	B-1
19.73396	st. dev	7	448	C-3
17	min	6	4	D-1
86	max	5	037999	F- 14
19	count	4	03	
		3	02	
		2	289	
		1	7	

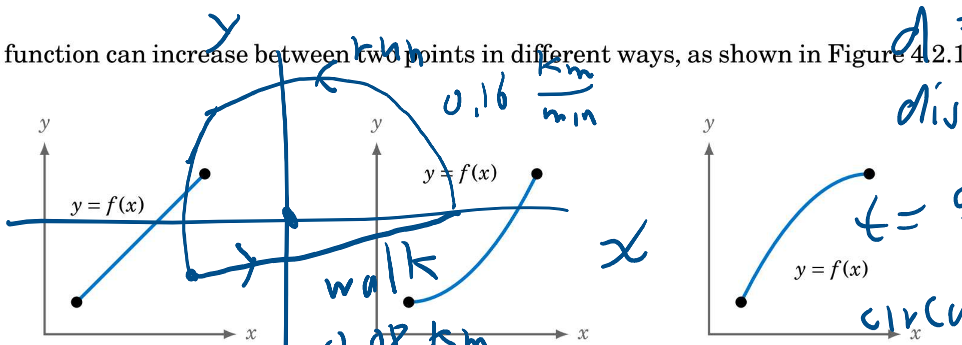
4.1

17. A certain jogger can run 0.16 km/min, and walks at half that speed. If he runs along a circular trail with circumference 50 km and then—before completing one full circle—walks back straight across to his starting point, what is the maximum time he can spend on the run/walk?

$d = rt$

distance = (rate)(time)

A function can increase between two points in different ways, as shown in Figure 4.2.1.



(a) $f'' = 0$: straight (b) $f'' > 0$: concave up (c) $f'' < 0$: concave down

Figure 4.2.1 Increasing function f: $f' > 0$, different signs for f''

$t = \frac{d}{r}$

circumference = 50 km

arc length $s = r\theta$

r = radius

θ = central angle

Think about it.

4.1: 7

7. The current I in a voltaic cell is

$$I = \frac{E}{R+r},$$

where E is the electromotive force and R and r are the external and internal resistance, respectively. Both E and r are internal characteristics of the cell, and hence can be treated as constants. The power P developed in the cell is $P = RI^2$. For which value of R is the power P maximized?

$$P(R) = RI^2 = R\left(\frac{E}{R+r}\right)^2$$

Maximize $P(R)$

2

Maximize $P(R)$

$$P(R) = \frac{R E^2}{(R+r)^2}$$

$$P'(R) = E^2 \frac{d}{dR}\left(\frac{R}{(R+r)^2}\right)$$

$$P'(R) = E^2\left(\frac{(R+r)^2 \frac{dR}{dR} - R\frac{d}{dR}(R+r)^2}{(R+r)^4}\right)$$

$$P'(R) = E^2\left(\frac{(R+r)^2 - R(2)(R+r)(1)}{(R+r)^4}\right)$$

$$P'(R) = \frac{E^2(R+r)(R+r-2R)}{(R+r)^4}$$

$$\boxed{P'(R) = \frac{E^2(-R+r)}{(R+r)^3}}$$

Not defined if $R+r=0 \iff R=-r$ impossible (by physics)

$$P'(R) = 0 = \frac{E^2(-R+r)}{(R+r)^3}$$

$$\Rightarrow E^2(-R+r) = 0$$

$$\Rightarrow -E^2R + E^2 r = 0$$

$$E^2 R = E^2 r$$

$\boxed{R = r}$ critical point

is this a global max?

2nd deriv test

$$\frac{d}{dR}\left(\frac{E^2(-R+r)}{(R+r)^3}\right)$$

$$\frac{d}{dR}\left(\frac{\quad}{(R+r)^3} \right)$$

$$= \frac{E^2\left((R+r)^3(-1) - (-R+r)(3)(R+r)^2 \right)}{(R+r)^6}$$

check if numerator is pos or neg
when $R = r$

$(2r)^3(-1) - (-r+r)(3)(2r)^2$

$-8r^3 < 0$

$r > 0$

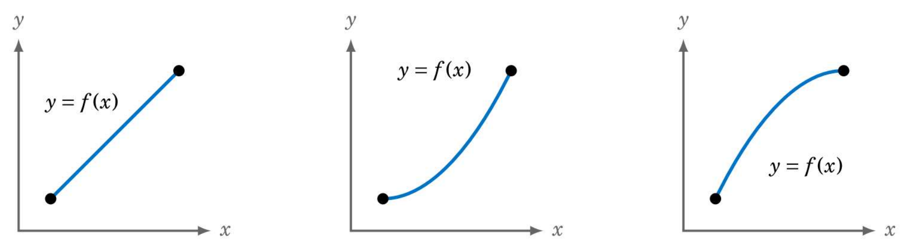
$\therefore$ local max

no other critical point

$\therefore$ global max

4.2

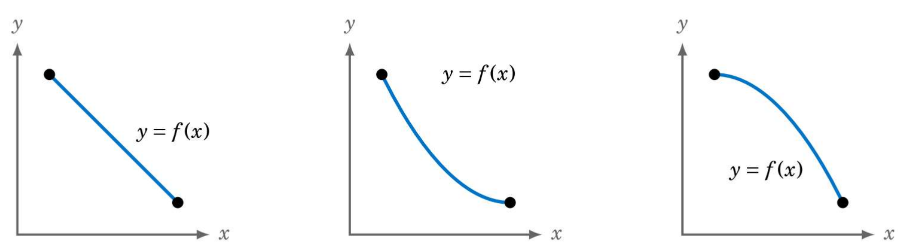
A function can increase between two points in different ways, as shown in Figure 4.2.1.



(a) $f'' = 0$: straight **(b)** $f'' > 0$: concave up **(c)** $f'' < 0$: concave down

Figure 4.2.1 Increasing function f: $f' > 0$, different signs for f''

informal definition: concave up - curve holds water
concave down - curve spills water



(a) $f'' = 0$ **(b)** $f'' > 0$: concave up **(c)** $f'' < 0$: concave down

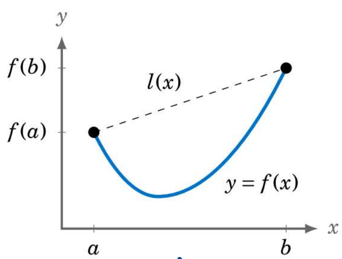
Figure 4.2.2 Decreasing function f: $f' < 0$, different signs for f''

Supplied

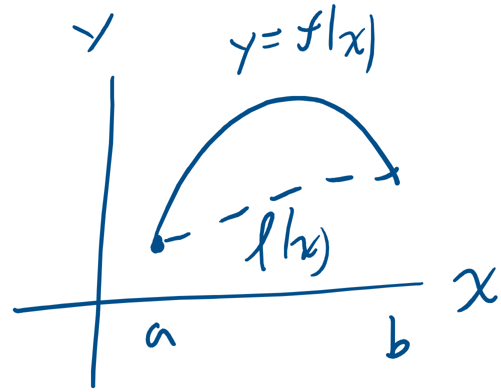
Concavity Theorem: Suppose that f is a twice-differentiable function on $[a, b]$. Then:

(a) If $f''(x) > 0$ on (a, b) then $f(x)$ is below the line $l(x)$ joining the points $(a, f(a))$ and $(b, f(b))$ for all x in (a, b).

(b) If $f''(x) < 0$ on (a, b) then $f(x)$ is above the line $l(x)$ joining the points $(a, f(a))$ and $(b, f(b))$ for all x in (a, b).

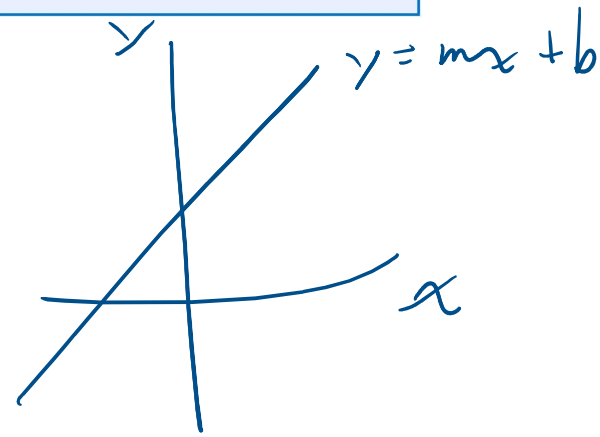


$f''(x) > 0$ on (a,b)



Memorize

A function f has an **inflection point** at $x = c$ if the concavity of f changes around $x = c$. That is, the function goes from concave up to concave down, or vice versa.



Let $f(x) = mx + b$

$f'(x) = m$

$f''(x) = 0$ all x

no inflection point

$\forall x \; f''(x) \not> 0$

$\forall x \; f''(x) \not< 0$

$\therefore$ concavity never change

(a) $f(x) = x^3$: inflection point at $x = 0$ **(b)** $f(x) = x^4$: non-inflection point at $x = 0$

Figure 4.2.3 Inflection vs non-inflection point at $x = 0$ with $f''(0) = 0$

Memorize

First Derivative Test: For a continuous function f on an interval I, let $x = c$ be a number in I such that $f(c)$ is defined, and either $f'(c) = 0$ or $f'(c)$ does not exist. Then:

(a) If $f'(x)$ changes from negative to positive around $x = c$ then f has a local minimum at $x = c$.

(b) If $f'(x)$ changes from positive to negative around $x = c$ then f has a local maximum at $x = c$.

Supplied but not used in this class

N

N^{th} Derivative Test: A function f with continuous derivatives of all orders up to and including $n > 1$ at $x = c$ has either a local minimum, local maximum or inflection point at $x = c$ if and only if

$$f^{(k)}(c) = 0 \text{ for } k = 1, 2, \ldots, n-1 \quad \text{and} \quad f^{(n)}(c) \neq 0$$

(i.e. the n^{th} derivative is the first nonzero derivative at $x = c$). If so, then:

(a) If $n > 1$ is even and $f^{(n)}(c) > 0$ then f has a local minimum at $x = c$.

(b) If $n > 1$ is even and $f^{(n)}(c) < 0$ then f has a local maximum at $x = c$.

(c) If $n > 1$ is odd then f has an inflection point at $x = c$.

4.2

For Exercises 1-8 sketch the graph of the given function. Find all local maxima and minima, inflection points, where the function is increasing or decreasing, where the function is concave up or concave down, and indicate any asymptotes.

2. $f(x) = x^3 - 3x^2 + 1$

$\boxed{f'(x) = 3x^2 - 6x}$ Defined for all x

$3x^2 - 6x = 0$

solve for x

$x^2 - 2x = 0$

$x(x-2) = 0$

$\boxed{x = 0, 2}$ critical points

$\boxed{f''(x) = 6x - 6}$

$f''(0) = 6(0) - 6 = -6 < 0$ local max $f(0) = 1$

$f''(2) = 6(2) - 6 = 6 > 0$ local min $f(2) = -3$

down

$f''(2) = 6(2) - 6 - \ldots$

concave up $6x - 6 > 0$

$6x > 6$

$x > 1$

concave down $6x - 6 < 0$

$6x < 6$

$x < 1$

inflection point $6x - 6 = 0$

$6x = 6$

$\boxed{x = 1}$ $f(1) = -1$

increase $f'(x) > 0$

$3x^2 - 6x > 0$

$x^2 - 2x > 0$

$x(x-2) > 0$

decrease

$3^2x - 6x < 0$

$\vdots$

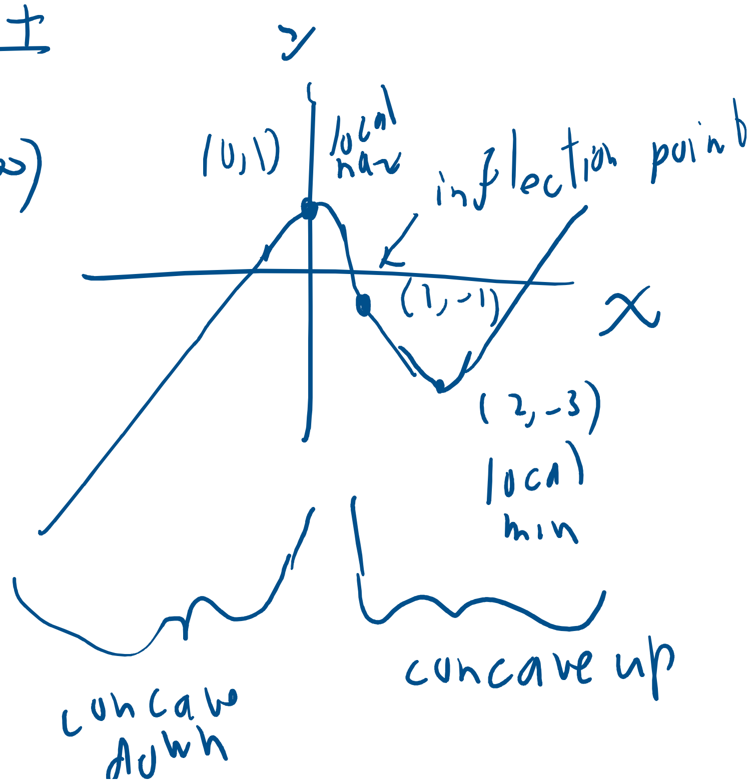
$x(x-2) < 0$

$(0, 2)$

solve $x(x-2) = 0$

$x = 0, 2$

increase $(-\infty, 0) \cup (2, \infty)$



concave
down