10-22-25 MTH 263

4 Applications of Derivatives

4.1 Optimization

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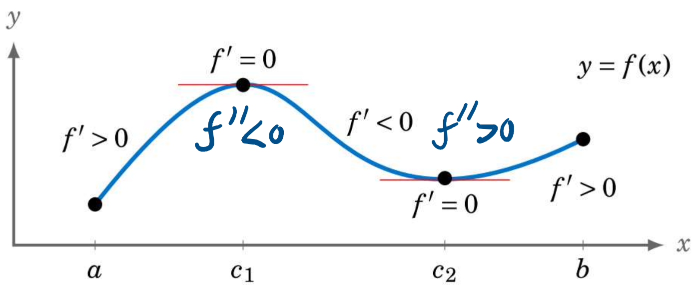
Took exam 2

Memorize

A function f has a **global maximum** at $x = c$ if $f(c) \geq f(x)$ for all x in the domain of f. Similarly, f has a **global minimum** at $x = c$ if $f(c) \leq f(x)$ for all x in the domain of f. Say that f has a **local maximum** at $x = c$ if $f(c) \geq f(x)$ for all x "near" c, i.e. for all x such that $|x - c| < \delta$ for some number $\delta > 0$. Likewise, f has a **local minimum** at $x = c$ if $f(c) \leq f(x)$ for all x such that $|x - c| < \delta$ for some number $\delta > 0$.

global = absolute
local = relative



Definition: a critical point is where f'(x) is not defined or f'(x) = 0.
Critical points are candidates for local max and local min.

f' goes from positive to zero to negative
$\Rightarrow$ f' is decreasing
$\Rightarrow f'' < 0$

Memorize

Second Derivative Test: Let $x = c$ be a critical point of f (i.e $f'(c) = 0$). Then:
(a) If $f''(c) > 0$ then f has a local minimum at $x = c$.

(b) If $f''(c) < 0$ then f has a local maximum at $x = c$.

(c) If $f''(c) = 0$ then the test fails.

Possibly useful

$f'' > 0$	$f'' < 0$	$f'' = 0$
local min.	local max.	test fails

Memorize

How to find a global maximum or minimum
Suppose that f is defined on an interval I. There are two cases:

1. **The interval I is closed**: The global maximum of f will occur either at an interior local maximum or at one of the endpoints of I whichever of these points provides the largest value of f will be where the global maximum occurs. Similarly, the global minimum of f will occur either at an interior local minimum or at one of the endpoints of I; whichever of these points provides the smallest value of f will be where the global minimum occurs.

2. **The interval I is not closed and has only one critical point**: If the only critical point is a local maximum then it is a global maximum. If the only critical point is a local minimum then it is a global minimum.

Let $f(x) = x^3 + 2x + 1$
on the interval $[-10, 10)$
Find global max, min

$f'(x) = 3x^2 + 2$ defined on $[-10, 10]$

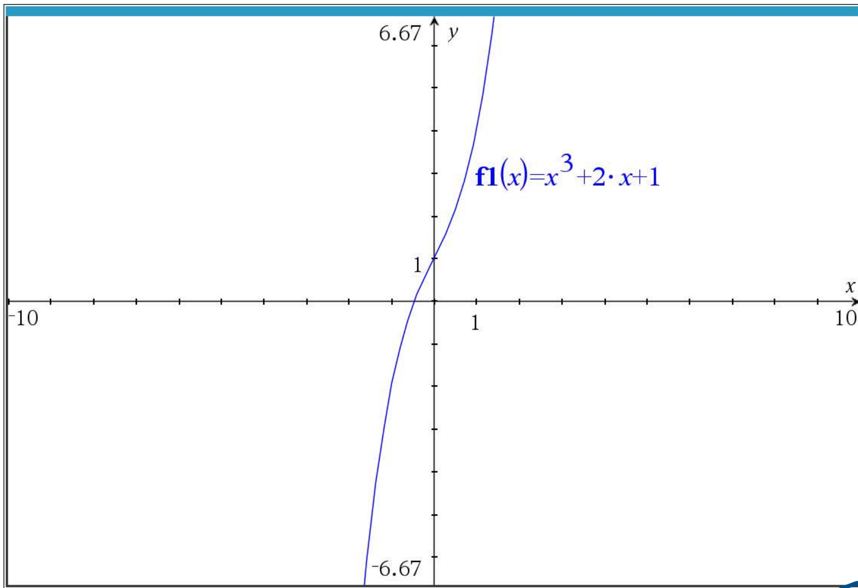
critical point: $3x^2 + 2 = 0$

$3x^2 = -2$

$x^2 = -\frac{2}{3}$

$x = \pm i\sqrt{\frac{2}{3}}$

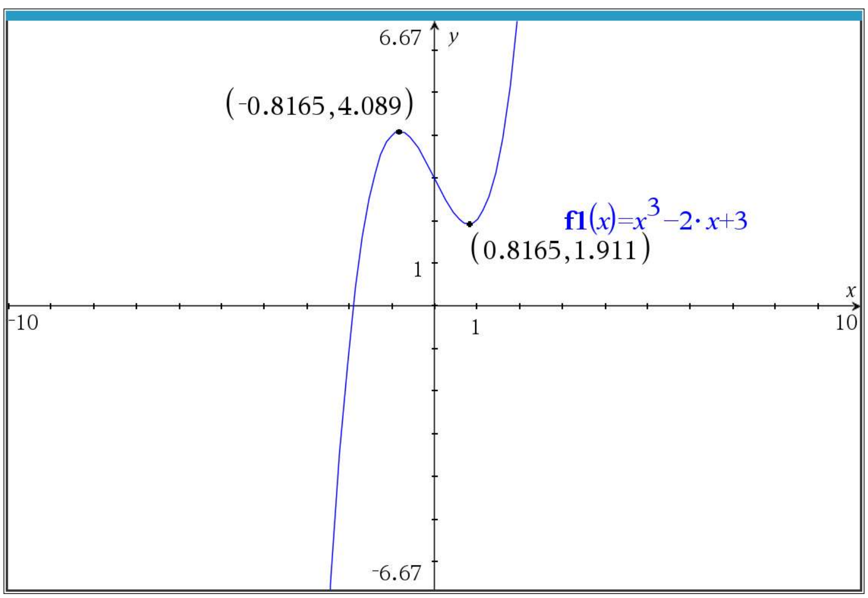
No critical point



$f(-10) = (-10)^3 + 2(-10) + 1 = -1000 - 19 = \boxed{-1019}$ global min

$f(10) = 10^3 + 2(10) + 1 = \boxed{1021}$ global max

Now, find global max and min on [-2,2]



$f'(x) = 3x^2 - 2 = 0$

$3x^2 = 2$

$\Rightarrow x^2 = \frac{2}{3}$

$\Rightarrow x = \pm\sqrt{\frac{2}{3}}$

Sqrt(2/3)=0.816496580927726

$f''(x) = 6x$

$f''(x) = 6x$

$f''\left(-\sqrt{\frac{2}{3}}\right) = 6\left(-\sqrt{\frac{2}{3}}\right) < 0$

$\Rightarrow$ local max

$f''\left(\sqrt{\frac{2}{3}}\right) = 6\left(\sqrt{\frac{2}{3}}\right) > 0 \Rightarrow$ local min

find global max, min on $[-2, 2]$

$f(-2) = (-2)^3 - 2[-2] + 3$

$= -8 + 4 + 3$

$\boxed{f(-2) = -1}$ global min

$f(2) = 2^3 - 2(2) + 3$

$\boxed{f(2) = 7}$ global max

$f\left(-\sqrt{\frac{2}{3}}\right) = \left(-\sqrt{\frac{2}{3}}\right)^3 - 2\left(-\sqrt{\frac{2}{3}}\right) + 3$

$f\left(-\sqrt{\frac{2}{3}}\right) \approx$ (-sqrt(2/3))^3+2*sqrt(2/3)+3=4.088662107903634 $\approx \boxed{4.09}$

$f\left(\sqrt{\frac{2}{3}}\right) \approx$ (sqrt(2/3))^3-2*sqrt(2/3)+3=1.911337892096365 $x \approx \boxed{1.91}$

Now, change the interval to [-1,1].
The local max and local min are the same as above.

$f(-1) = (-1)^3 - 2(-1) + 3 = -1 + 2 + 3 = \boxed{4} < 4.09$

$f(1) = 1^3 - 2(1) + 3 = \boxed{2} > 1.91$

Now, the interior critical points become global max and global min