10-20-25 MTH 263

3.6 Differentials
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Exam 2, Wednesday, 10/22/25, 2.1-2.4, 3.1 - 3.6

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14 class meetings before final exam
10 textbook sections
10/14=0.7143 section per class meeting

3.6:9

9. The *continuity relation* for an ideal gas is

$$\frac{PM}{\sqrt{T}} = \text{constant}$$

where P and T are the pressure and temperature, respectively, of the gas, and M is the *Mach number*. Show that

$$\frac{dP}{P} + \frac{dM}{M} = \frac{dT}{2T}.$$

$$\frac{PM}{\sqrt{T}} = k = \text{constant}$$

$$PM = k\sqrt{T}$$

$$d(PM) = d(k\sqrt{T})$$

$$P\,dM + M\,dP = k\,d(T^{\frac{1}{2}})$$

$$P\,dM + M\,dP = (k)\left(\tfrac{1}{2}\right)T^{-\frac{1}{2}}\,dT$$

$$\frac{\cancel{P}\,dM}{M\cancel{P}} + \frac{\cancel{M}\,dP}{\cancel{M}P} = \frac{k}{MP}\left(\frac{1}{2}\right)T^{-\frac{1}{2}}\,dT$$

$$\frac{dM}{M} + \frac{dP}{P} = \left(\frac{k}{2MP\sqrt{T}}\right)dT$$

$$\stackrel{?}{=} T$$

$$\boxed{\frac{PM}{\sqrt{T}} = k} = \frac{\cancel{PM}}{2\sqrt{T}\,\cancel{MP}\sqrt{T}}\,dT$$

$$\boxed{\frac{dM}{M} + \frac{dP}{P} = \frac{dT}{2T}}$$

$$\frac{PM}{\sqrt{T}} = k$$

$$\ln\left(\frac{PM}{\sqrt{T}}\right) = \ln(k)$$

$$\ln\left(\frac{\sqrt{T}}{\frac{PM}{\sqrt{T}}}\right) = \ln(k)$$

$$\ln P + \ln M - \ln\left(T^{\frac{1}{2}}\right) = \ln(k)$$

$$\ln P + \ln M - \tfrac{1}{2}\ln T = \ln(k)$$

$$d(\ln P) + d\ln(M) - \tfrac{1}{2}d\ln T = d(\ln k)$$

$$\frac{dP}{P} + \frac{dM}{M} - \frac{1}{2}\frac{dT}{T} = 0$$

$$\frac{dP}{P} + \frac{dM}{M} = \frac{dT}{2T}$$

3.6: 3

3. Show that $d\left(\tan^{-1}(y/x)\right) = \dfrac{x\,dy\ -\ y\,dx}{x^2\ +\ y^2}$

Remember $\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$

$$d\left(\tan^{-1}\left(\frac{y}{x}\right)\right)$$

$$= \frac{1}{1+\frac{y^2}{x^2}}\ d\left(\frac{y}{x}\right)$$

$$= \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{x\,dy - y\,dx}{x^2}\right)$$

$$= \frac{1}{\frac{x^2+y^2}{x^2}}\left(\frac{x\,dy - y\,dx}{x^2}\right)$$

$$= \frac{\cancel{x^2}}{x^2+y^2}\left(\frac{x\,dy - y\,dx}{\cancel{x^2}}\right)$$

$$= \frac{x\,dy - y\,dx}{x^2+y^2} \quad \checkmark$$

Squeeze Theorem: Suppose that for some functions f, g and h there is a number $x_0 \ge 0$ such that

$$g(x) \le f(x) \le h(x) \text{ for all } x > x_0$$

and that $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} h(x) = L$. Then $\lim_{x\to\infty} f(x) = L$.

Similarly, if $g(x) \le f(x) \le h(x)$ for all $x \ne a$ in some interval I containing a, and if $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} f(x) = L$.

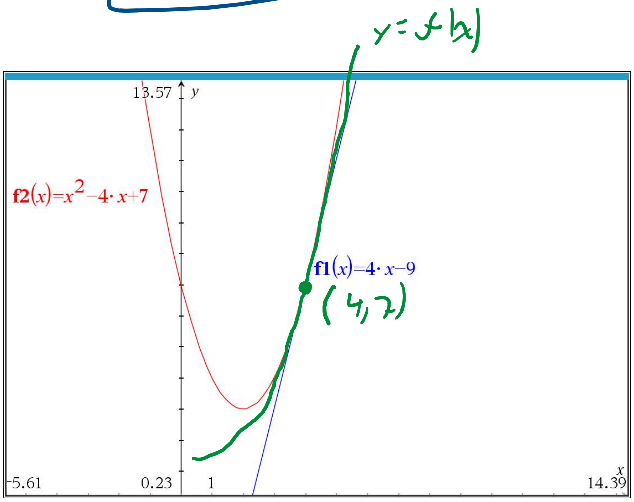
Guichard

6. For all $x \ge 0$, $4x - 9 \le f(x) \le x^2 - 4x + 7$. Find $\lim_{x\to 4} f(x)$.

$$\lim_{x\to 4}(4x-9) \le \lim_{x\to 4} f(x) \le \lim_{x\to 4}(x^2-4x+7)$$

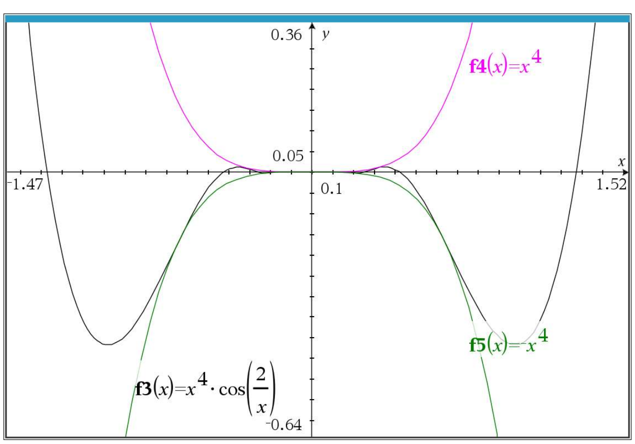
$$7 \le \lim_{x\to 4} f(x) \le 7$$

$$\boxed{\therefore \lim_{x\to 4} f(x) = 7}$$



Guichard

8. Use the Squeeze Theorem to show that $\lim_{x\to 0} x^4 \cos(2/x) = 0$.



$-x^4 \le x^4 \cos\left(\frac{2}{x}\right) \le x^4$ is this true?

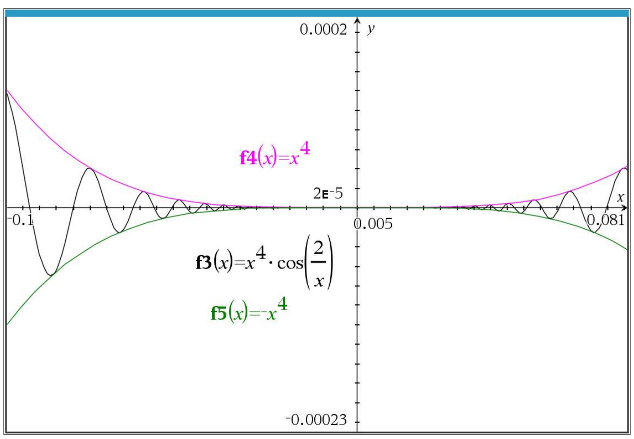
$-x^4 \le x^4 \cos\left(\frac{2}{x}\right) \le x^4$ is this true?

$\Longleftrightarrow \; -1 \le \cos\left(\frac{2}{x}\right) \le 1$ True

$\lim_{x \to 0} (-x^4) \le \lim_{x \to 0} x^4 \cos\left(\frac{2}{x}\right) \le \lim_{x \to 0} (x^4)$

$0 \le \lim_{x \to 0} x^4 \cos\left(\frac{2}{x}\right) \le 0$

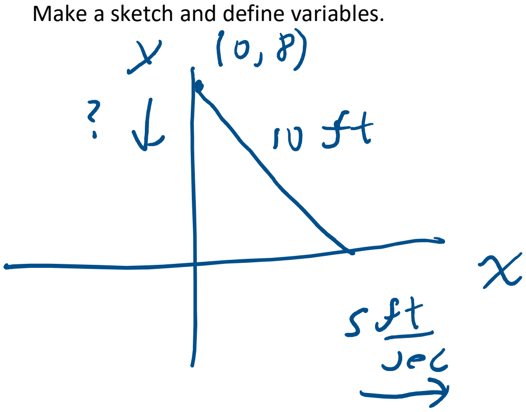
$\therefore \lim_{x \to 0} x^4 \cos\left(\frac{2}{x}\right) = 0$ ✓



3.5: 4

4. A 10-ft ladder is leaning against a wall on level ground. If the bottom of the ladder is dragged away from the wall at the rate of 5 ft/s, how fast will the top of the ladder descend at the instant when it is 8 ft from the ground?

Make a sketch and define variables.



Let $y(t)$ = height of ladder (ft) at time t (sec)

Let $x(t)$ = distance of base (ft) at time t (sec)

Find $\frac{dy}{dt}$ when $y(t) = 8$ ft

$\frac{dx}{dt} = 5 \frac{ft}{sec}$

$dt \qquad sec$

$$x^2 + y^2 = (10\ ft)^2 = 100\ ft^2$$

$$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(100\ ft^2)$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0\ \frac{ft^2}{sec}$$

$$2(6\ ft)\left(5\ \frac{ft}{sec}\right) + 2(8\ ft)\frac{dy}{dt} = 0\ \frac{ft^2}{sec}$$

$$x^2 = 100 - y^2$$

$$x^2 = 100 - 64$$

$$x^2 = 36$$

$$\boxed{x = 6}$$

$$60\ \frac{ft^2}{sec} + 16\ ft\ \frac{dy}{dt} = 0$$

$$(16\ ft)\frac{dy}{dt} = -60\ \frac{ft^2}{sec}$$

$$\frac{dy}{dt} = -\frac{60}{16}\ \frac{\frac{ft^2}{sec}}{ft}$$

$$\boxed{\frac{dy}{dt} = -3.75\ \frac{ft}{sec}}$$

The top of the ladder descends at a rate of 3.75 ft/sec.

$$\lim_{x\to\infty} \frac{x^2}{x} \left[\frac{\infty}{\infty}\right] \underset{L}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}(x^2)}{\frac{d}{dx}(x)}$$

$$= \lim_{x\to\infty} \frac{2x}{1} = \lim_{x\to\infty} 2x = \infty \text{ (dne)}$$

does not exist

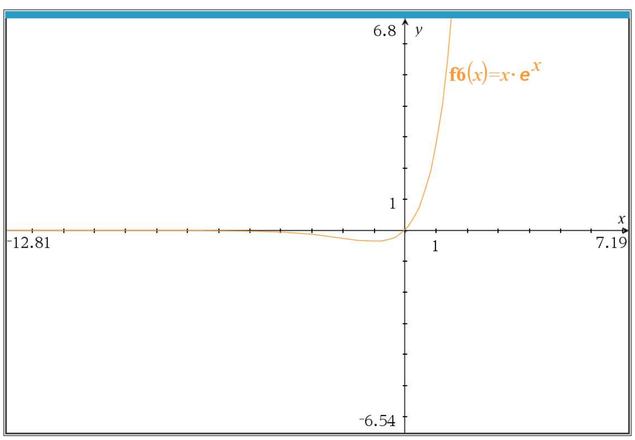
$x \qquad x$

$$\lim_{x\to-\infty} x e^{x} = \lim_{x\to-\infty} \frac{e^{x}}{\frac{1}{x}} \quad \overset{L}{=}$$

$$[-\infty\cdot 0] \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to-\infty} \frac{\frac{d}{dx}(e^{x})}{\frac{d}{dx}(x^{-1})} = \lim_{x\to\infty} \frac{e^{x}}{-x^{-2}}$$

$$= \lim_{x\to-\infty} -x^{2} e^{x} = ?$$



It appears that

$$\lim_{x\to-\infty} xe^{x} = 0$$

$$\ln\left(\lim_{x\to-\infty} xe^{x}\right)$$

$$= \lim_{x\to+\infty} \ln(xe^{x})$$

$$= \lim_{x\to-\infty} [\ln x + \ln e^{x}]$$

$$= \lim [\ln x + x]$$

This is not defined form

$= \lim_{x \to -\infty} [\ln x + x]$

This is not defined form negative values of x.

Gemini

2.5 Flash

Upgrade

Find lim as x approaches minus infinity of x*e^x.

Analysis

The limit of xe^x as x approaches $-\infty$ is 0.

This limit is an indeterminate form of the type $(-\infty) \cdot 0$, which can be solved by rewriting the expression and applying **L'Hôpital's Rule**.

Derivation

The limit is:

$$\lim_{x \to -\infty} xe^x$$

First, we rewrite the expression to get the indeterminate form $\frac{\infty}{\infty}$:

$$\lim_{x \to -\infty} xe^x = \lim_{x \to -\infty} \frac{x}{e^{-x}}$$

As $x \to -\infty$, the numerator $x \to -\infty$, and the denominator $e^{-x} \to e^{\infty} = \infty$. The form is $\frac{-\infty}{\infty}$.

Now, we apply **L'Hôpital's Rule**, which states that for a limit of the form $\frac{\infty}{\infty}$, the limit is equal to the limit of the ratio of the derivatives of the numerator and the denominator:

$$\lim_{x \to -\infty} \frac{\frac{d}{dx}(x)}{\frac{d}{dx}(e^{-x})} = \lim_{x \to -\infty} \frac{1}{-e^{-x}}$$

Finally, we evaluate the limit:

$$\lim_{x \to -\infty} \frac{1}{-e^{-x}}$$

As $x \to -\infty$, the exponent $-x \to \infty$, so $e^{-x} \to \infty$. Therefore, the denominator $-e^{-x} \to -\infty$.

$$\frac{1}{-\infty} = 0$$

Therefore,

$$\lim_{x \to -\infty} xe^x = 0$$

This result is a specific case of a more general principle in limits: **exponentials grow (or decay) faster than any polynomial**. In this case, e^x approaches 0 much faster than x approaches $-\infty$, causing the entire product to approach 0.