3.5 Related Rates
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3.6 Differentials
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4 Applications of Derivatives
4.1 Optimization
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Exam 2, Wednesday, 10/22/25, 2.1-2.4, 3.1 - 3.6

3.6
Memorize

For a differentiable function $f(x)$, the **differential** of $f(x)$ is

$$df = f'(x)\,dx \qquad (3.7)$$

where dx is an infinitesimal change in x.

Memorize

Let f and g be differentiable functions, and let c be a constant. Then:

(a) $d(c) = 0$

(b) $d(cf) = c\,df$ (Constant Multiple Rule)

(c) $d(f+g) = df + dg$ (Sum Rule)

(d) $d(f-g) = df - dg$ (Difference Rule)

(e) $d(fg) = f\,dg + g\,df$ (Product Rule)

(f) $d\left(\frac{f}{g}\right) = \frac{g\,df - f\,dg}{g^2}$ (Quotient Rule)

(g) $d(f^n) = nf^{n-1}\,df$ (Power Rule)

(h) $d(f(g)) = \frac{df}{dg}\,dg$ (Chain Rule)

$$d(\ln u) = \frac{du}{u}$$

Prove $\frac{d}{dx}\left(\ln u(x)\right) = \frac{1}{u(x)}\frac{du}{dx}$

Multiply each term by dx

$$d(\ln(u(x))) = \frac{du}{u(x)}$$

$$d(\ln(u)) = \frac{du}{u}$$

4.1
Memorize

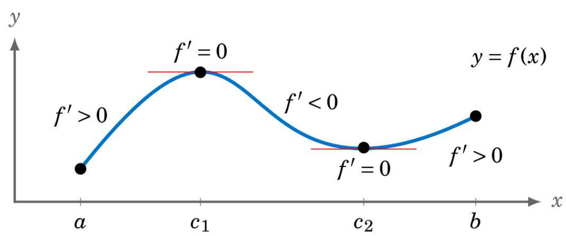
A function f has a **global maximum** at $x = c$ if $f(c) \geq f(x)$ for all x in the domain of f. Similarly, f has a **global minimum** at $x = c$ if $f(c) \leq f(x)$ for all x in the domain of f. Say that f has a **local maximum** at $x = c$ if $f(c) \geq f(x)$ for all x "near" c, i.e. for all x such that $|x - c| < \delta$ for some number $\delta > 0$. Likewise, f has a **local minimum** at $x = c$ if $f(c) \leq f(x)$ for all x such that $|x - c| < \delta$ for some number $\delta > 0$.

Memorize

Second Derivative Test: Let $x = c$ be a critical point of f (i.e $f'(c) = 0$). Then:
(a) If $f''(c) > 0$ then f has a local minimum at $x = c$.

(b) If $f''(c) < 0$ then f has a local maximum at $x = c$.

(c) If $f''(c) = 0$ then the test fails.



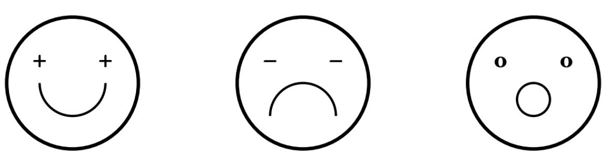
around $x=c_1$

f' goes from pos to 0 to neg
$\Rightarrow f'$ is decreasing
$\Rightarrow f'' < 0$
local max

around $x=c_2$

f' goes from neg to 0 to pos
$\Rightarrow f'$ is increasing
$\Rightarrow f'' > 0$
local min

Helpful



$f'' > 0$ local min.	$f'' < 0$ local max.	$f'' = 0$ test fails

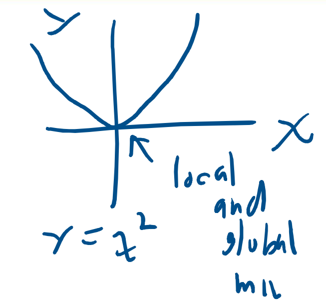
Memorize concept

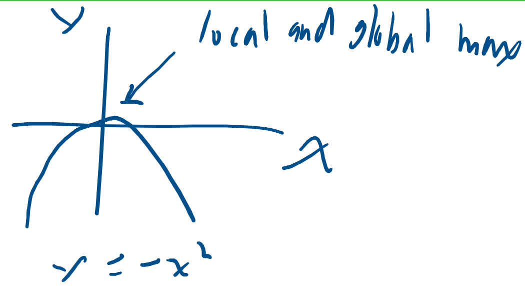
How to find a global maximum or minimum
Suppose that f is defined on an interval I. There are two cases:

1. **The interval I is closed**: The global maximum of f will occur either at an interior local maximum or at one of the endpoints of I whichever of these points provides the largest value of f will be where the global maximum occurs.
   Similarly, the global minimum of f will occur either at an interior local minimum or at one of the endpoints of I; whichever of these points provides the smallest value of f will be where the global minimum occurs.

2. **The interval I is not closed and has only one critical point**: If the only critical point is a local maximum then it is a global maximum. If the only critical point is a local minimum then it is a global minimum.

2. The interval I is not closed and has only one critical point. If the only critical point is a local maximum then it is a global maximum. If the only critical point is a local minimum then it is a global minimum.





Note: a critical point of the function f(x) is where the derivative = 0 or does not exist. (mentioned in the next section.)

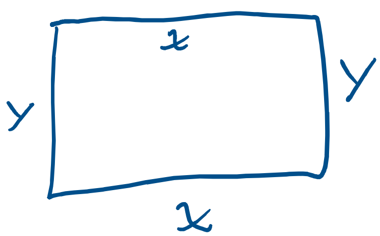
Guichard

Summary—Steps to solve an optimization problem.

1. Decide what the variables are and what the constants are, draw a diagram if appropriate, understand clearly what it is that is to be maximized or minimized.
2. Write a formula for the function for which you wish to find the maximum or minimum.
3. Express that formula in terms of only one variable, that is, in the form $f(x)$.
4. Set $f'(x) = 0$ and solve. Check all critical values and endpoints to determine the extreme value.

Guichard 6.1

2. Find the dimensions of the rectangle of largest area having fixed perimeter 100.



Let x and y be the dimensions of a rectangle.

Let A = area of rectangle

Find x, y to maximize A

$A(x,y) = xy$

$2(x+y) = 100$

$2x + 2y = 100$

$x + y = 50$

$\boxed{y = 50 - x}$

$\boxed{A(x) = x(50-x)}$

$A(x) = 50x - x^2$

$0 \le x$ because we can't have a negative dimension

$x \le 50$ because $x = 50$ $\Rightarrow 2x = 100$ and $y = 0$

$\therefore 0 \le x \le 50$

Find critical point,

$A'(x) = 50 - 2x$ all x, i.e. A' defined everywhere

set $A'(x) = 0$, solve for x

$A'(x) = 50 - 2x$ all x, ie A' defined everywhere

set $A'(x) = 0$, solve for x

$50 - 2x = 0 \Rightarrow 50 = 2x$

$2x = 50$

$\boxed{x = 25}$

$y = 50 - 25$

$\boxed{y = 25}$

$A(25) = (25)^2 = 625$

$A(0) = 0(50) = 0$

$A(50) = 50(0) = 0$

$\boxed{\therefore \text{ global max} = 625 \text{ at } x = 25, y = 25}$

global min $= 0$ at $x = 50$, $y = 0$

$y = 50$, $x = 0$

Alternative method (ignore geometric constraint)

$A(x)$ defined on $(-\infty, \infty)$

use 2nd derivative test

$A''(x) = \frac{d}{dx}(50 - 2x)$

$= -2 < 0$

$\therefore$ local max at $x = 25$

this is the only critical point

so it is also a global max

Guichard

DEFINITION 6.4.3 Let $y = f(x)$ be a differentiable function. We define a new independent variable dx, and a new dependent variable $dy = f'(x)\,dx$. Notice that dy is a function both of x (since $f'(x)$ is a function of x) and of dx. We say that dx and dy are **differentials**. □

Let $\Delta x = x - a$ and $\Delta y = f(x) - f(a)$. If x is near a then Δx is small. If we set $dx = \Delta x$ then

$$dy = f'(a)\,dx \approx \frac{\Delta y}{\Delta x}\Delta x = \Delta y.$$

Thus, dy can be used to approximate Δy, the actual change in the function f between a and x. This is exactly the approximation given by the tangent line:

$$dy = f'(a)(x - a) = f'(a)(x - a) + f(a) - f(a) = L(x) - f(a).$$

While $L(x)$ approximates $f(x)$, dy approximates how $f(x)$ has changed from $f(a)$. Figure 6.4.2 illustrates the relationships.

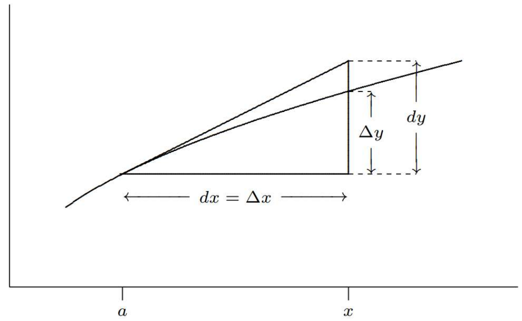


Figure 6.4.2 Differentials.

3.6

2. Find the differential df of $f(x) = \sin^2(x^2)$.

$$\frac{df}{dx} = 2\sin(x^2)\cos(x^2)(2x)$$

$$\frac{df}{dx} = 4x\sin(x^2)\cos(x^2)$$

$$\boxed{df = 4x\sin(x^2)\cos(x^2)\,dx}$$

$$df = 2x(2\sin(x^2)\cos(x^2))\,dx$$

$$\boxed{df = 2x\sin(2x^2)\,dx}$$

Gemini agrees

3.6

4. Given $y^2 - xy + 2x^2 = 3$, find dy.

$$\frac{d}{dx}(y^2) - \frac{d}{dx}(xy) + \frac{d}{dx}(2x^2) = \frac{d}{dx}(3)$$

$$2y\frac{dy}{dx} - x\frac{dy}{dx} - y\frac{dx}{dx} + 4x = 0$$

$$2y\frac{dy}{dx} - x\frac{dy}{dx} - y\frac{dx}{dx} + 4x = 0$$

$$\frac{dy}{dx}(2y-x) = y-4x$$

$$\frac{dy}{dx} = \frac{y-4x}{2y-x}$$

$$\boxed{dy = \left(\frac{y-4x}{2y-x}\right)dx}$$

4.1

2. Prove that for $0 \le p \le 1$, $p(1-p) \le \frac{1}{4}$.

Let $f(p) = p(1-p) = p - p^2$

$f(p)$ is defined on the closed, bounded interval $[0,1]$

Find global max of $f(p)$

$\frac{df}{dp} = 1-2p$, defined for all p

$$1-2p=0$$

$$2p=1$$

$$\boxed{p=\frac{1}{2}}$$

$f\left(\frac{1}{2}\right) = \frac{1}{2}\left(1-\frac{1}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$ global max

$\left.\begin{array}{l} f(0) = 0(1-0) = 0 \\ f(1) = 1(1-1) = 0 \end{array}\right\}$ global min

$$f(1) = 1(1-1) = 0)$$

$$\to f(p) \leq \frac{1}{4} \text{ all } p \in [0,1]$$