10-06-25 MTH 263

3.2 Limits: Formal Definition
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3.3 Continuity
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3.2: 13

For Exercises 1-18 evaluate the given limit.

13. $\lim_{x\to 0} \frac{\ln(1-x) - \sin^2 x}{1 - \cos^2 x}$ $= \lim_{x\to 0} f(x)$

$\frac{\ln(1-0) - \sin^2 0}{1 - \cos^2 0}$ $\frac{\ln(1) - 0}{1-1} = \frac{0-0}{0} = \frac{0}{0}$ not defined

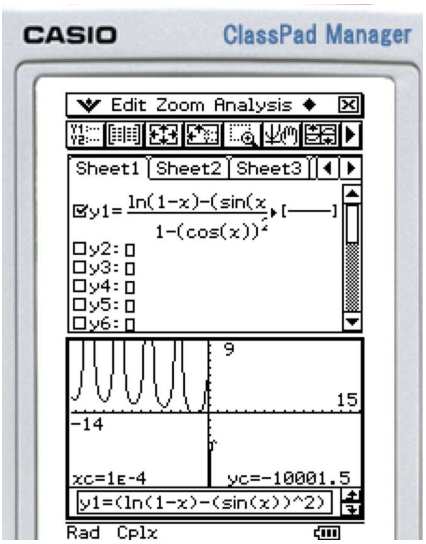
$\therefore f(0)$ is not defined

$\left[\frac{0}{0}\right]$ $\overset{L}{=} \lim_{x\to 0} \frac{\frac{d}{dx}\left(\ln(1-x) - \sin^2 x\right)}{\frac{d}{dx}\left(1 - \cos^2 x\right)}$

$= \lim_{x\to 0} \frac{\frac{1}{1-x}(-1) - 2\sin x \cos x}{0 - 2\cos x(-\sin x)}$

$= \lim_{x\to 0} \frac{\frac{-1}{1-x} - 2\sin x\cos x}{2\sin x \cos x}$

$= \frac{\frac{-1}{1-0} - 2\sin 0\cos 0}{2\sin 0\cos 0} = \frac{-1-0}{0} = -\infty$ (dne)



The graph supports, but does not prove, our calculation.

Find $\lim_{x \to 1} (x-1)(x+2)$ and verify with formal definition

$$\lim_{x \to 1} (x-1)(x+2) = (1-1)(0+2) = 0$$

Let $\varepsilon > 0$ Find $\delta > 0$ such that

$$0 < |x-1| < \delta \Rightarrow |(x-1)(x+2) - 0| < \varepsilon$$

$$|(x-1)(x+2) - 0| < \varepsilon$$

$$\Leftrightarrow |(x-1)(x+2)| < \varepsilon$$

$$\Leftrightarrow |x-1||x+2| < \varepsilon$$

Let $\delta_1 = 1$
use this to find a bound on $|x+2|$

$$|x-1| < 1$$

$$-1 < x-1 < 1$$

$$-1+3 < x-1+3 < 1+3$$

$$2 < x+2 < 4$$

$$\Rightarrow \boxed{|x+2| < 4}$$

$2 < x + 2 <$

$\Rightarrow \boxed{|x+2| < 4}$

want $|x-1|\,|x+2| < \varepsilon$

we have $|x-1|\,|x+2| < |x-1|\,4$

we would be done if $4|x-1| < \varepsilon$

$\Leftrightarrow \boxed{|x-1| < \frac{\varepsilon}{4}}$

To satisfy both constraints, let $\delta = \min\left(1, \frac{\varepsilon}{4}\right)$

e.g. if $\varepsilon = 8$

then $\delta = \min\left(1, \frac{8}{4}\right) = \min(1, 2) = 1$

if $\varepsilon = \frac{1}{100}$, then $\delta = \min\left(1, \frac{1}{400}\right) = \frac{1}{400}$

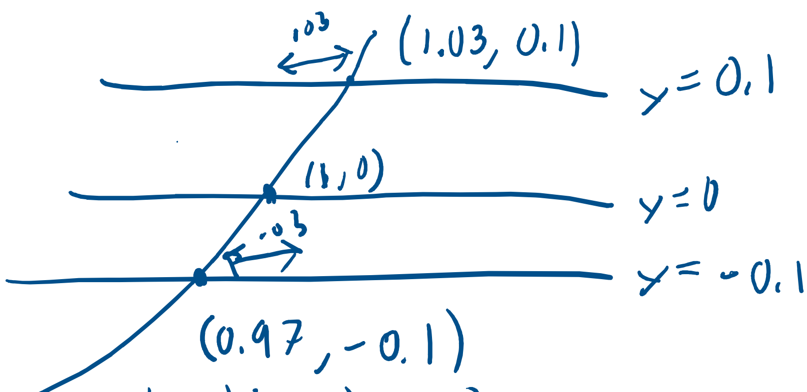
You are responsible for applying the formal definition of a limit to linear functions.

Calculator example for a given epsilon.

Let $\varepsilon = 0.1$

Find δ such that

$|x-1| < \delta \Rightarrow |(x-1)(x+2) - 0| < 0.1$



$y = (x-1)(x+2) = x^2 + x - 2$

$\therefore$ let $\delta = 0.03$

$\therefore$ Let $\delta = 0.03$

$$\delta = \min\left(1, \frac{\varepsilon}{4}\right) = \min\left(1, \frac{.1}{4}\right)$$
$$= \min(1, .025)$$
$$= .025 \approx 0.03$$

3.3
Memorize

A function f is **continuous** at $x = a$ if

$$\lim_{x \to a} f(x) = f(a)\,. \tag{3.4}$$

A function is continuous on an interval I if it is continuous at every point in the interval. For a closed interval $I = [a, b]$, a function f is continuous on I if it is continuous on the open interval (a, b) and if $\lim_{x \to a+} f(x) = f(a)$ (i.e. f is **right continuous** at $x = a$) and $\lim_{x \to b-} f(x) = f(b)$ (i.e. f is **left continuous** at $x = b$). A function is **discontinuous** at a point if it is not continuous there. A continuous function is one that is continuous over its entire domain.

this is equivalent to

$\lim_{x \to a^-} f(x)$ exists

$\lim_{x \to a^+} f(x)$ exists

$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \Rightarrow \lim_{x \to a} f(x)$ exists

$f(a)$ is defined

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$

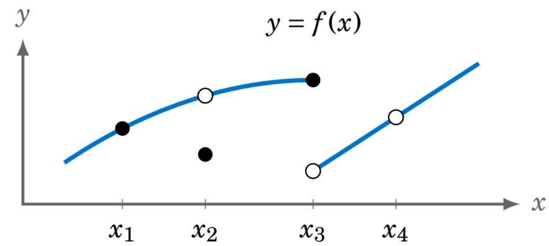


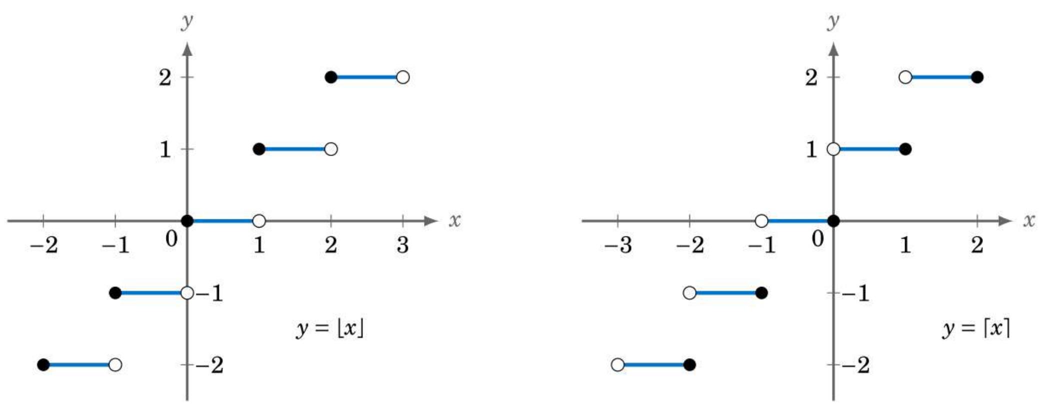
Figure 3.3.1 Continuous at x_1, discontinuous at x_2, x_3 and x_4

Example 3.22

The *floor function* $\lfloor x \rfloor$ is defined as

$$\lfloor x \rfloor \;=\; \text{the largest integer less than or equal to } x \,.$$

In other words, $\lfloor x \rfloor$ rounds a non-integer down to the previous integer, and integers stay the same. For example, $\lfloor 0.1 \rfloor = 0$, $\lfloor 0.9 \rfloor = 0$, $\lfloor 0 \rfloor = 0$, and $\lfloor -1.3 \rfloor = -2$. The graph of $\lfloor x \rfloor$ is shown in Figure 3.3.2(a).



(a) Floor function $\lfloor x \rfloor$ **(b)** Ceiling function $\lceil x \rceil$

Figure 3.3.2 Floor and ceiling functions

Memorize

If f is a continuous function and $\lim\limits_{x\to a} g(x)$ exists and is finite, then:

$$f\left(\lim_{x\to a} g(x)\right) \;=\; \lim_{x\to a} f(g(x)) \tag{3.5}$$

The same relation holds for one-sided limits.

Memorize theorem

Every differentiable function is continuous.

Proof: If a function f is differentiable at $x = a$ then $f'(a) = \lim\limits_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists, so

$$\lim_{x\to a}\,(f(x)-f(a)) \;=\; \lim_{x\to a}\,(f(x)-f(a))\cdot\frac{x-a}{x-a} \;=\; \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\lim_{x\to a}\,(x-a) \;=\; f'(a)\cdot 0 \;=\; 0$$

which means that $\lim\limits_{x\to a} f(x) \;=\; f(a)$, i.e. f is continuous at $x = a$. ✓

Note: the converse is false.
For example $|x|$ has no derivative at x = 0, but it is continuous on $(-\infty, \infty)$

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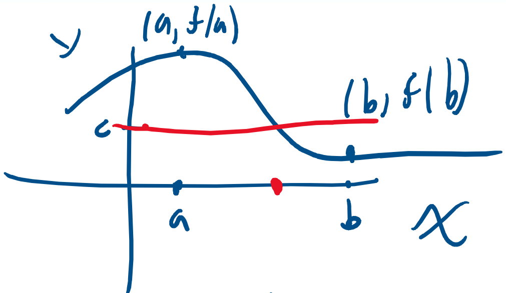
Memorize

Extreme Value Theorem: If f is a continuous function on a closed interval $[a, b]$ then f attains both a maximum value and a minimum value on that interval.

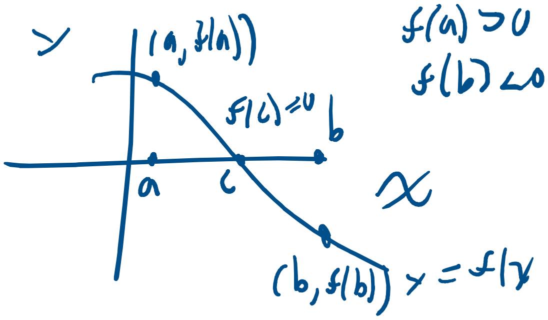
Intermediate Value Theorem: If f is a continuous function on a closed interval $[a, b]$ then f attains every value between $f(a)$ and $f(b)$.



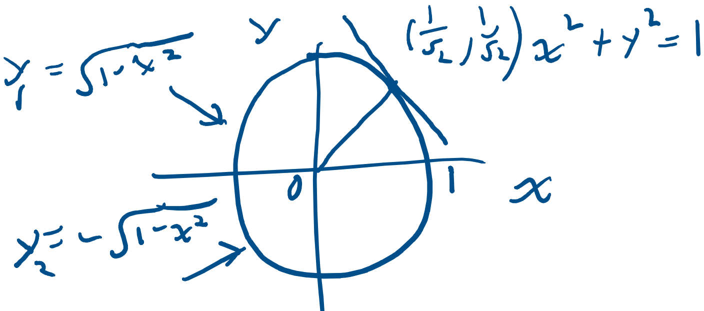
Let $c \in (a, b)$

special case
Let $f(a)$, $f(b)$ have opposite signs

$f(a) > 0$
$f(b) < 0$



3.4 implicit differentiation.



Find $\frac{dy}{dx}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$\frac{dy_1}{dx} = \frac{d}{dx}\sqrt{1-x^2} = \frac{d}{dx}\left(1-x^2\right)^{\frac{1}{2}}$$

$$= \frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}(-2x)$$

$$= \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)$$

$$\boxed{\frac{dy_1}{dx} = -\frac{x}{\sqrt{1-x^2}}}$$

$$\frac{dy_1}{dx}\left(\frac{1}{\sqrt{2}}\right) = -\frac{\frac{1}{\sqrt{2}}}{\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2}} = -\frac{\frac{1}{\sqrt{2}}}{\sqrt{1-\frac{1}{2}}} = \frac{-\frac{1}{\sqrt{2}}}{\sqrt{\frac{1}{2}}} = -1$$

Now, use implicit differentiation

$$x^2 + y^2 = 1$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(1)$$

$$2x + 2y\frac{dy}{dx} = 0$$

solve algebraically for $\frac{dy}{dx}$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{2x}{2y}$$

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

$$\frac{dy}{dx}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = -1$$

3.4

For Exercises 1-9, use implicit differentiation to find $\frac{dy}{dx}$.

2. $xy = (x+y)^3$

$$\frac{d}{dx}(xy) = \frac{d}{dx}(x+y)^3$$

$$x\frac{dy}{dx} + y\frac{dx}{dx} = 3(x+y)^2\frac{d}{dx}(x+y)$$

$$x\frac{dy}{dx} + y\frac{dx}{dx} = 3(x+y)^2 \frac{d}{dx}(x+y)$$

$$x\frac{dy}{dx} + y = 3(x+y)^2\left(1 + \frac{dy}{dx}\right)$$

$$x\frac{dy}{dx} = 3(x+y)^2 + 3(x+y)^2\frac{dy}{dx} - y$$

$$\left(x - 3(x+y)^2\right)\frac{dy}{dx} = 3(x+y)^2 - y$$

$$\boxed{\frac{dy}{dx} = \frac{3(x+y)^2 - y}{x - 3(x+y)^2}}$$