3 Topics in Differential Calculus

3.1 Tangent Lines

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3.2 Limits: Formal Definition

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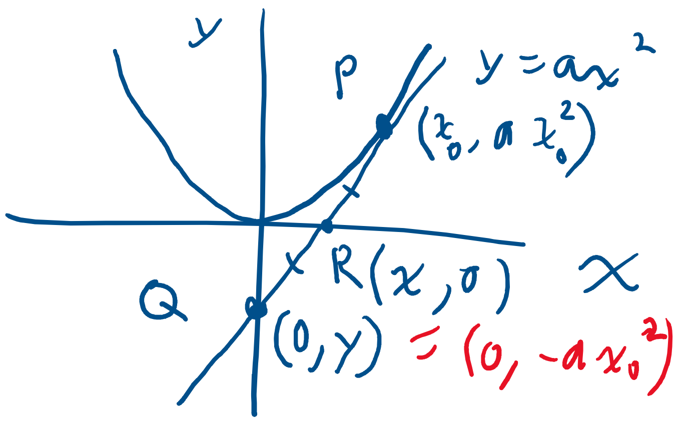
3.1: 27

27. For a constant $a > 0$, let P be a point on the curve $y = ax^2$, and let Q be the point where the tangent line to the curve at P intersects the y-axis. Show that the x-axis bisects the line segment $\overline{PQ}$.



Let R be the intersection of the tangent line with the x-axis

show $\overline{PR} = \overline{QR}$

Find x, y

· Find equation of tangent line

$y - y_0 = f'(x_0)(x - x_0)$

$y_0 = a x_0^2$

$f'(x) = \frac{d}{dx}(ax^2) = 2ax$

$f'(x_0) = 2 a x_0$

$y - y_0 = 2 a x_0 (x - x_0)$

$y = 2 a x_0 x - 2 a x_0^2 + y_0$

$y = 2 a x_0 x - 2 a x_0^2 + a x_0^2$

$y = 2ax_0 x - 2ax_0^2 + ax_0^2$

$\boxed{y = 2ax_0 x - ax_0^2}$

$2ax_0 x - ax_0^2 = 0$

$2ax_0 x = ax_0^2$

$x = \frac{ax_0^2}{2ax_0}$

$\boxed{x = \frac{x_0}{2}}$

0 is halfway between ax_0^2 and $-ax_0^2$

$\boxed{\therefore R = \left(\frac{x_0}{2}, \frac{y_0}{2}\right)}$ satisfies the midpoint formula

Alternative approach

use coordinates to show $\overline{PR} = \overline{QR}$ using the distance formula

3.2

Memorize

A real number L is the limit of $f(x)$ as x approaches a if the values of $f(x)$ can be made *arbitrarily* close to L by picking values of x *sufficiently* close to a.

Memorize

Let L and a be real numbers. Then L is the **limit** of a function $f(x)$ as x approaches a, written as

$$\lim_{x \to a} f(x) = L,$$

if for any given number $\epsilon > 0$, there exists a number $\delta > 0$, such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

such that $|x-a| < \delta \implies |f(x) - L| < \epsilon$

$|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

such that $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

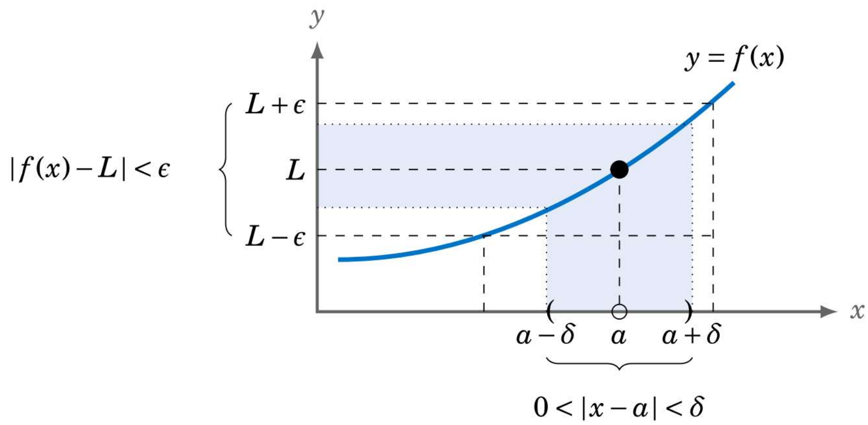


Figure 3.2.1 $\lim_{x \to a} f(x) = L$

Memorize

Call L the **right limit** of a function $f(x)$ as x approaches a, written as

$$\lim_{x \to a+} f(x) = L ,$$

if $f(x)$ approaches L as x approaches a for values of x larger than a.
Call L the **left limit** of a function $f(x)$ as x approaches a, written as

$$\lim_{x \to a-} f(x) = L ,$$

if $f(x)$ approaches L as x approaches a for values of x smaller than a.

Memorize

The limit of a function exists if and only if both its right limit and left limit exist and are equal:

$$\lim_{x \to a} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to a-} f(x) = L = \lim_{x \to a+} f(x)$$

Find $\lim_{x \to 3} (2x - 5)$

Then verify the limit using the formal definition

$\lim (2x - 5) = 2(3) - 5 = 6 - 5 = \boxed{1}$

$$\lim_{x\to 3}(2x-5) = 2(3)-5 = 6-5 = \boxed{1}$$

Let $\varepsilon > 0$

Find $\delta > 0$ such that

in general, $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

Here $0 < |x-3| < \delta \Rightarrow |(2x-5)-1| < \varepsilon$

$$|(2x-5)-1| < \varepsilon$$

$$\Longleftrightarrow |2x-6| < \varepsilon$$

$$\Longleftrightarrow |2||x-3| < \varepsilon$$

$$\Longleftrightarrow 2|x-3| < \varepsilon$$

$$\Longleftrightarrow |x-3| < \frac{\varepsilon}{2}$$

Let $\boxed{\delta = \frac{\varepsilon}{2}}$ ← This is enough

$$\therefore \lim_{x\to 3}(2x-5) = 1$$

$\delta = \frac{\varepsilon}{2} \Rightarrow$

$$|x-3| < \frac{\varepsilon}{2}$$

$$\Rightarrow 2|x-3| < \varepsilon$$

$$\Rightarrow |2x-6| < \varepsilon$$

$$\Rightarrow |2x-5-1| < \varepsilon$$

$$\Rightarrow |(2x-5)-1| < \varepsilon$$

$\Rightarrow |2x - 5 - 1| < \varepsilon$

$\Rightarrow |(2x-5)-1| < \varepsilon$

Memorize

For a real number a, the limit of a function $f(x)$ **equals infinity** as x approaches a, written as

$$\lim_{x \to a} f(x) = \infty \,,$$

if $f(x)$ grows without bound as x approaches a, i.e. $f(x)$ can be made larger than any positive number by picking x sufficiently close to a:
For any given number $M > 0$, there exists a number $\delta > 0$, such that

$$f(x) > M \quad \text{whenever} \quad 0 < |x - a| < \delta \,.$$

Memorize

For a real number a, the limit of a function $f(x)$ **equals negative infinity** as x approaches a, written as

$$\lim_{x \to a} f(x) = -\infty \,,$$

if $f(x)$ grows negatively without bound as x approaches a, i.e. $f(x)$ can be made smaller than any negative number by picking x sufficiently close to a:
For any given number $M < 0$, there exists a number $\delta > 0$, such that

$$f(x) < M \quad \text{whenever} \quad 0 < |x - a| < \delta \,.$$

Example 3.11

Evaluate $\lim\limits_{x \to 0} \frac{1}{x}$.

Solution: For $x \neq 0$ the function $f(x) = \frac{1}{x}$ is defined, and its graph is shown in Figure 3.2.6. As x approaches 0 from the right, $1/x$ approaches ∞, that is,

$$\lim_{x \to 0+} \frac{1}{x} = \infty \,.$$

As x approaches 0 from the left, $1/x$ approaches $-\infty$, that is,

$$\lim_{x \to 0-} \frac{1}{x} = -\infty \,.$$

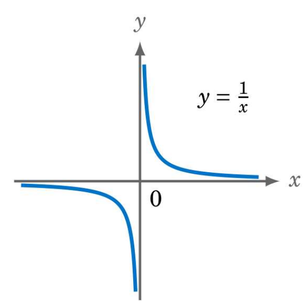


Figure 3.2.6

Since the right limit and the left limit are not equal, then $\lim\limits_{x \to 0} \frac{1}{x}$ does not exist . (dne)

Note that the y-axis (i.e. the line $x = 0$) is a vertical asymptote for $f(x) = \frac{1}{x}$.

$\lim\limits_{x \to 0^+} \left(\frac{1}{x}\right)$	∞
$\lim\limits_{x \to 0^-} \left(\frac{1}{x}\right)$	$-\infty$
$\lim\limits_{x \to 0} \left(\frac{1}{x}\right)$	undef

Memorize

For a real number L, the limit of a function $f(x)$ equals L as x approaches ∞, written as

$$\lim_{x\to\infty} f(x) = L\,,$$

if $f(x)$ can be made arbitrarily close to L for x sufficiently large and positive: For any given number $\epsilon > 0$, there exists a number $N > 0$, such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad x > N\,.$$

Memorize

For a real number L, the limit of a function $f(x)$ equals L as x approaches $-\infty$, written as

$$\lim_{x\to-\infty} f(x) = L\,,$$

if $f(x)$ can be made arbitrarily close to L for x sufficiently small and negative: For any given number $\epsilon > 0$, there exists a number $N < 0$, such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad x < N\,.$$

Memorize

$$\lim_{x\to\infty} x^n = \begin{cases} \infty & \text{for any real } n > 0 \\ 0 & \text{for any real } n < 0 \end{cases} \qquad \lim_{x\to-\infty} x^n = \begin{cases} \infty & \text{for } n = 2,4,6,8,\ldots \\ -\infty & \text{for } n = 1,3,5,7,\ldots \end{cases}$$

$$\lim_{x\to\infty} e^x = \infty \qquad \lim_{x\to-\infty} e^x = 0 \qquad \lim_{x\to\infty} e^{-x} = 0 \qquad \lim_{x\to-\infty} e^{-x} = \infty$$

$$\lim_{x\to\infty} \ln x = \infty \qquad \lim_{x\to 0+} \ln x = -\infty$$

Not on quiz or exam

Say that

$$f(x) = O(g(x)) \quad \text{as } x \to \infty,$$

spoken as "f is big O of g", if there exist positive numbers M and x_0 such that

$$|f(x)| \le M|g(x)| \quad \text{for all } x \ge x_0.$$

Memorize

L'Hôpital's Rule: If f and g are differentiable functions and

$f(x) \to \pm\infty$
and $g(x) \to \pm\infty$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty} \quad \text{or} \quad \frac{0}{0}$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}\,.$$

The number a can be real, ∞, or $-\infty$.

lim x = lim 1 = 1

$$\lim_{x \to \infty} \frac{x}{x} = \lim_{x \to \infty} 1 = 1$$

$$\lim_{x \to \infty} \frac{x}{x^2} = \lim_{x \to \infty} \left(\frac{1}{x}\right) = 0$$

$$\lim_{x \to \infty} \frac{x^2}{x} = \lim_{x \to \infty} x = \infty \quad (\text{dne})$$

Each of the above have the form $\frac{\infty}{\infty}$

Therefore, we call such expressions "indeterminant."

Now, calculate the limits using L'Hopital's rule.

$$\underset{\left[\frac{\infty}{\infty}\right]}{\lim_{x \to \infty} \frac{x}{x}} \overset{L}{=} \lim_{x \to \infty} \frac{\frac{dx}{dx}}{\frac{dx}{dx}} = \lim_{x \to \infty} \frac{1}{1} = \lim_{x \to \infty} (1) = 1$$

apply L'Hopital's rule

$$\underset{\left[\frac{\infty}{\infty}\right]}{\lim_{x \to \infty} \frac{x}{x^2}} \overset{L}{=} \lim_{x \to \infty} \frac{\frac{d}{dx}(x)}{\frac{d}{dx}(x^2)} = \lim_{x \to \infty} \frac{1}{2x} = 0$$

$$\underset{\left[\frac{\infty}{\infty}\right]}{\lim_{x \to \infty} \frac{x^2}{x}} \overset{L}{=} \lim_{x \to \infty} \frac{\frac{d}{dx}(x^2)}{\frac{d}{dx}(x)} = \lim_{x \to \infty} \frac{2x}{1} = \lim_{x \to \infty} 2x = \infty$$

0

∞

$g(x)$

$f(\ldots$

$g(x)^{\infty}$

$f(x)$

$f(x) \to \infty$
$g(x) \to 0$

What indeterminants are there besides infinity/infinity and 0/0?

There are **five other indeterminate forms** besides $\frac{\infty}{\infty}$ and $\frac{0}{0}$.

Indeterminate Forms

An indeterminate form is an expression that arises in the context of limits where the limit of the individual parts results in an ambiguous expression, and the value of the limit cannot be determined solely by substituting the value.

The seven indeterminate forms are:

1. $\frac{0}{0}$
2. $\frac{\infty}{\infty}$
3. $\infty - \infty$
4. $0 \cdot \infty$
5. 1^{∞}
6. 0^{0}
7. ∞^{0}

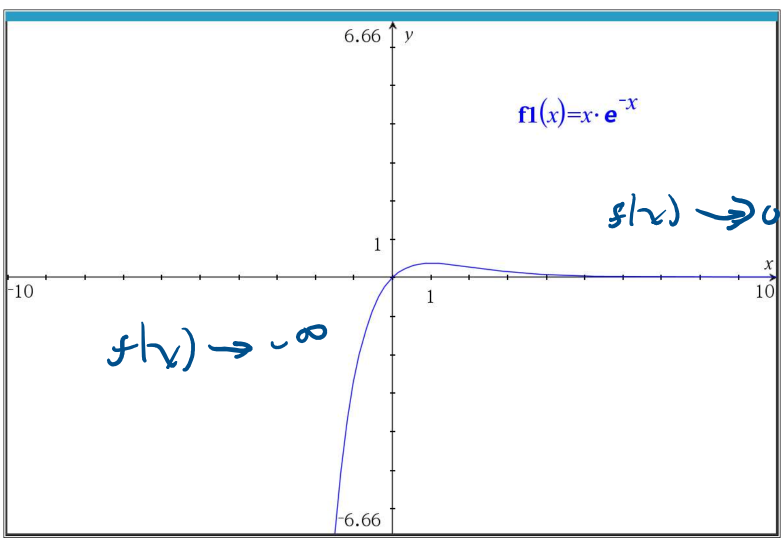
Transformation to $\frac{0}{0}$ or $\frac{\infty}{\infty}$

To evaluate limits involving these other indeterminate forms, one typically manipulates the expression algebraically, or uses logarithms, to transform it into one of the forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$, which can then be solved using **L'Hôpital's Rule**.

Indeterminate Form	Typical Transformation	Example
$\infty - \infty$	Combine fractions (find a common denominator), rationalize, or factor.	$\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{\sin x}\right)$
$0 \cdot \infty$	Rewrite as a fraction: $f(x)g(x) = \frac{f(x)}{1/g(x)}$ or $\frac{g(x)}{1/f(x)}$.	$\lim_{x\to\infty} x \cdot e^{-x}$
$1^\infty, 0^0, \infty^0$	Use logarithms or the identity $f(x)^{g(x)} = e^{g(x)\ln f(x)}$ to transform the exponent to a $0 \cdot \infty$ form, and then to a fractional form.	$\lim_{x\to 0}(1 + \sin x)^{1/x}$

$$\lim_{x\to\infty} x e^{-x} \; [\infty \cdot 0] = \lim_{x\to\infty} \frac{x}{e^x} \; \left[\frac{\infty}{\infty}\right] \overset{L}{=}$$

$$\lim_{x\to\infty} \frac{\frac{d}{dx}(x)}{\frac{d}{dx}(e^x)} = \lim_{x\to\infty} \frac{1}{e^x} = 0$$



Squeeze Theorem: Suppose that for some functions f, g and h there is a number $x_0 \geq 0$ such that

$$g(x) \leq f(x) \leq h(x) \text{ for all } x > x_0$$

and that $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} h(x) = L$. Then $\lim_{x\to\infty} f(x) = L$.

Similarly, if $g(x) \leq f(x) \leq h(x)$ for all $x \neq a$ in some interval I containing a, and if $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} f(x) = L$.

Example 3.21

Evaluate $\lim_{x\to\infty} \dfrac{\sin x}{x}$.

Solution: Since $-1 \le \sin x \le 1$ for all x, then dividing all parts of those inequalities by $x > 0$ yields

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x} \text{ for all } x > 0 \quad \Rightarrow \quad \lim_{x\to\infty} \frac{\sin x}{x} = 0$$

by the Squeeze Theorem, since $\lim_{x\to\infty} \dfrac{-1}{x} = 0 = \lim_{x\to\infty} \dfrac{1}{x}$.

$$\lim_{x\to\infty} \frac{\sin x}{x} = \frac{\text{dne}}{\infty}$$