2.3 The Exponential and Natural Logarithm Functions
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2.4 General Exponential and Logarithmic Functions
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3 Topics in Differential Calculus
3.1 Tangent Lines
page 71: 1, 5, 11, 15, 23, 27

2.3: 19

19. Suppose it takes 8 hours for 30% of a radioactive substance to decay. Find the half-life of the substance.

$A(t) = A_0 e^{kt}$

$A(t)$ = amount of radioactive material remaining at time t

A_0 = initial amount

$A(0) = A_0$

t_H = half-life = time to reduce $A(t)$ by $\frac{1}{2}$

$\boxed{A(8) = 0.7 A_0}$

Find t_H

$A(8) = A_0 e^{k(8)} = 0.7 A_0$

$\Rightarrow e^{8k} = 0.7$

solve for k

$\ln(e^{8k}) = \ln(0.7)$

$8k = \ln(0.7)$

$\boxed{k = \frac{\ln(0.7)}{8}}$

$A(t) = A_0 e^{\left(\frac{\ln(0.7)}{8}\right)t}$

$A(t_H) = A_0 e^{\left(\frac{\ln(0.7)}{8}\right)t_H} = \frac{A_0}{2}$

$$A(t_H) = A_0\, e^{\left(\frac{\ln(0.7)}{8}\right)t_H} = \frac{A_0}{2}$$

$$\Rightarrow \quad e^{\left[\frac{\ln(0.7)}{8}\right]t_H} = \frac{1}{2}$$

$$\Rightarrow \ln\left[e^{\left(\frac{\ln(0.7)}{8}\right)t_H}\right] = \ln\left(\frac{1}{2}\right)$$

$$= \frac{\ln(0.7)}{8}\, t_H = \ln\left(\frac{1}{2}\right)$$

$$\Rightarrow t_H = \frac{8\ln\left(\frac{1}{2}\right)}{\ln(0.7)}$$

$$\frac{8\cdot\ln\left(\frac{1}{2}\right)}{\ln(0.7)} \qquad 15.5469$$

The half-life is approximately 15.5 hours

2.3: 17

For Exercises 15-18, use logarithmic differentiation to find $\frac{dy}{dx}$.

17. $y = x^{\sin x}$

$$\ln x = \ln\left(x^{\sin x}\right)$$

$$\ln y = (\sin x)\ln(x)$$

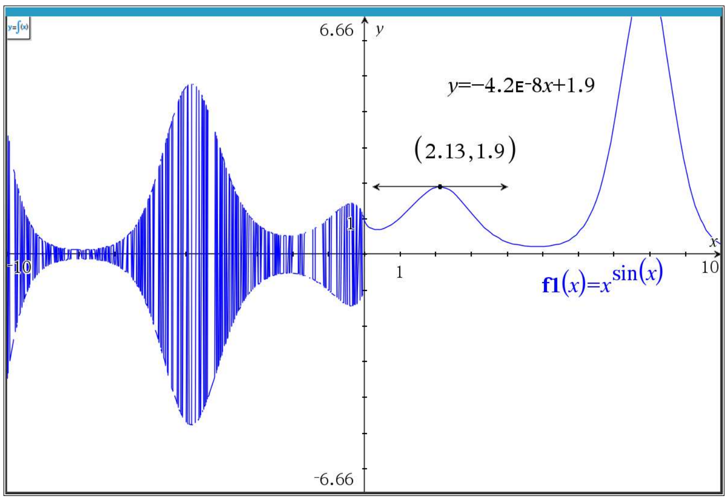
$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\Big((\sin x)(\ln x)\Big)$$

$$\left(\frac{1}{y}\right)\frac{dy}{dx} = (\sin x)\frac{d}{dx}(\ln x) + \ln(x)\frac{d}{dx}(\sin x)$$

$$\left(\frac{1}{y}\right)\frac{dy}{dx} = (\sin x)\left(\frac{1}{x}\right) + \ln(x)\cos x$$

$$\frac{dy}{dx} = y\left[\frac{\sin x}{x} + \ln(x)\cos x\right]$$

$$\boxed{\frac{dy}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \ln(x)\cos x\right]}$$



3.1
Memorize

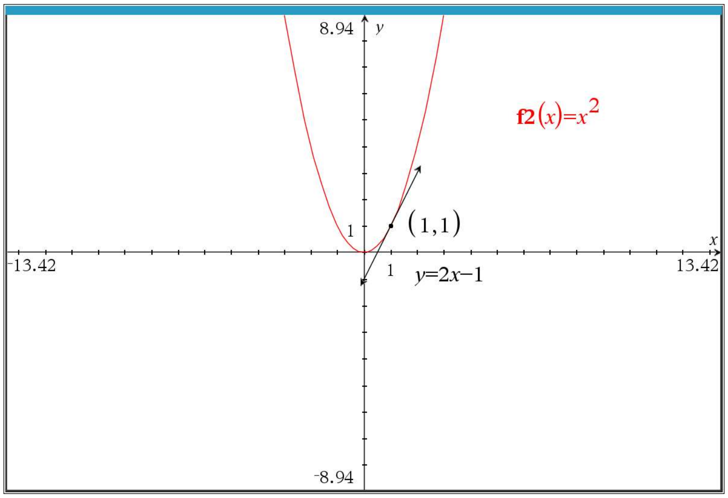
For a curve $y = f(x)$ that is differentiable at $x = a$, the **tangent line** to the curve at the point $P = (a, f(a))$ is the unique line through P with slope $m = f'(a)$. P is called the **point of tangency**. The equation of the tangent line is thus given by:

$$y - f(a) = f'(a)\cdot(x - a) \qquad (3.1)$$

TI inspire

$\frac{d}{dx}\left(x^2\right)$	$2\cdot x$	← not on TI-84
$\frac{d}{dx}\left(x^2\right)\vert x=1$	2	← TI-84

TI nderiv(x^2,x,1)



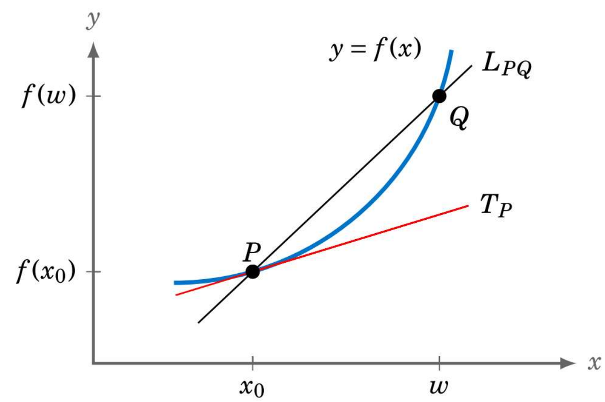
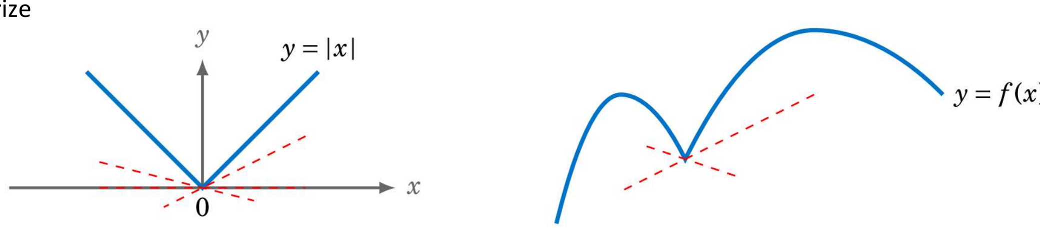


Figure 3.1.5 Secant line L_{PQ} approaching the tangent line T_P as $Q \to P$

memorize



(a) $f(x) = |x|$ **(b)** Curve with a cusp

Figure 3.1.6 Nonsmooth curves: no tangent line at the nonsmooth points

Supplied (not on exam)

The tangent line to a curve $y = f(x)$ makes an angle $\phi(x)$ with the positive x-axis, given by

$$\phi(x) = \tan^{-1} f'(x)\,. \tag{3.2}$$

Memorize

The equation of the normal line to a curve $y = f(x)$ at a point $P = (a, f(a))$ is

$$y - f(a) = -\frac{1}{f'(a)}(x - a) \quad \text{if } f'(a) \neq 0 \tag{3.3}$$

The equation of the normal line to a curve $y = f(x)$ at a point $P = (a, f(a))$ is

$$y - f(a) = -\frac{1}{f'(a)} \cdot (x - a) \quad \text{if } f'(a) \neq 0. \tag{3.3}$$

If $f'(a) = 0$, then the normal line is vertical and is given by $x = a$.

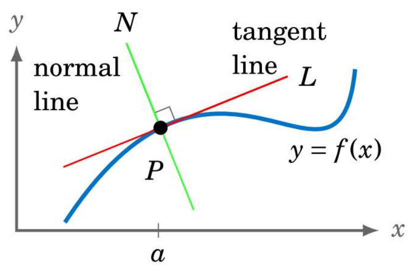


Figure 3.1.8 Normal line N

3.1

For Exercises 1-12, find the equation of the tangent line to the curve $y = f(x)$ at $x = a$.

6. $f(x) = e^x$; at $x = 0$

Do this first analytically, then graphically. Sketch a labeled graph.

Hint: slope of tangent line = derivative of the function at that point

point of tangency

$f(0) = e^0 = 1$

point of tangency is $(0,1)$

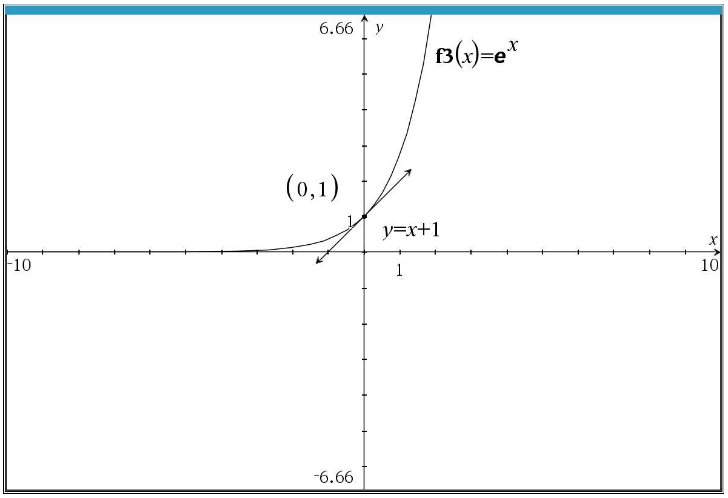
$f'(x) = \frac{d}{dx}(e^x) = e^x$

$f'(0) = e^0 = 1$

$y - 1 = 1(x - 0)$

$y - 1 = x$

$\boxed{y = x + 1}$



Quiz 4 Your Name MTH 263 Write each problem. Closed notes, need calculator.

1. **13.** Show that $\frac{d}{dx}(\ln(kx)) = \frac{1}{x}$ for all constants $k > 0$.

$$\frac{d}{dx}(\ln(kx))$$

$$= \frac{1}{kx}\,\frac{d}{dx}(kx) = \left(\frac{1}{kx}\right)(k) = \boxed{\frac{1}{x}}$$

$$\frac{d}{dx}(\ln(kx)) = \frac{d}{dx}(\ln k + \ln x)$$

$$= 0 + \frac{1}{x} = \boxed{\frac{1}{x}}$$

2. Use log differentiation to find $\frac{dy}{dx}$ for $y = \frac{(x^2+2)(3-x)}{x^2}$.

$$\ln y = \ln\left[\frac{(x^2+2)(3-x)}{x^2}\right]$$

$$\ln y = \ln(x^2+2) + \ln(3-x) - \ln(x^2)$$

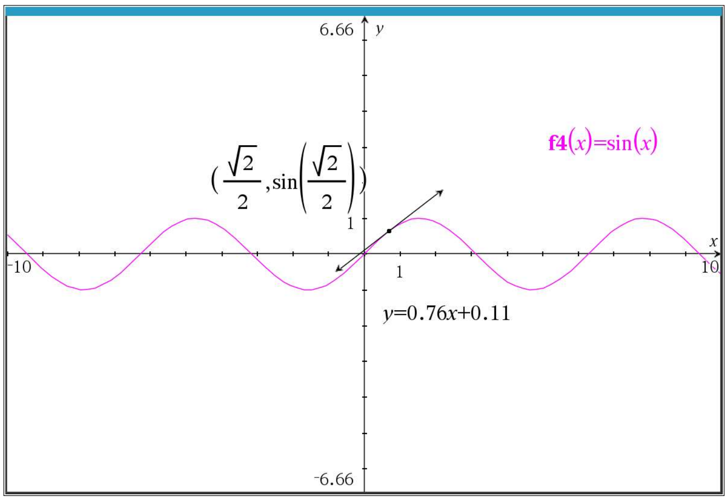
$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(\ln(x^2+2) + \ln(3-x) - \ln(x^2)\right)$$

$$\left(\frac{1}{y}\right)\frac{dy}{dx} = \frac{1}{x^2+2}(2x) + \frac{1(-1)}{3-x} - \left(\frac{1}{x^2}\right)(2x)$$

$$\boxed{\frac{dy}{dx} = \frac{(x^2+2)(3-x)}{x^2}\left(\frac{2x}{x^2+2} - \frac{1}{3-x} - \frac{2}{x}\right)}$$

$$\boxed{\frac{dy}{dx} = \frac{(x^2+2)(3-x)}{x^2}\left(\frac{2x}{x^2+2} - \frac{1}{3-x} - \frac{2}{x}\right)}$$

3. Find the equation of the tangent line to $f(x) = \sin(x)$ at the point $x = \frac{1}{\sqrt{2}}$. First manually, then graphically. Sketch a labeled graph. Round answer to nearest hundredth.



$$y - f\left(\frac{1}{\sqrt{2}}\right) = f'\left(\frac{1}{\sqrt{2}}\right)\left(x - \frac{1}{\sqrt{2}}\right)$$

$$f'(x) = \frac{d}{dx}\sin x = \cos x$$

$$f'\left(\frac{1}{\sqrt{2}}\right) = \cos\left(\frac{1}{\sqrt{2}}\right)$$

$$y - \sin\left(\frac{1}{\sqrt{2}}\right) = \cos\left(\frac{1}{\sqrt{2}}\right)\left(x - \frac{1}{\sqrt{2}}\right)$$

```
sin(1/√2)
                0.6496369391
cos(1/√2)
                0.7602445971
```

```
0.65-0.76*0.71
                0.1104
```

$$y - 0.65 = 0.76(x - 0.71)$$

$$y = 0.76x + 0.65 - (0.76)(0.71)$$

$$\boxed{y = 0.76x - 0.11}$$ tangent line