09-24-25 MTH 263

2.2 Trigonometric Functions and Their Inverses
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2.3 The Exponential and Natural Logarithm Functions
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20 remaining class meetings before final exam
18 textbook sections
About 1 section per class meeting

2.2: 4

A

For Exercises 1-16, find the derivative of the given function $y = f(x)$.

4. $y = \cos(\tan x)$

$$\frac{dy}{dx} = -\sin(\tan x)\frac{d}{dx}(\tan x)$$

$$= -\sin(\tan x)(\sec^2 x)$$

2.2: 5

5. $y = \tan^{-1}(x/3)$

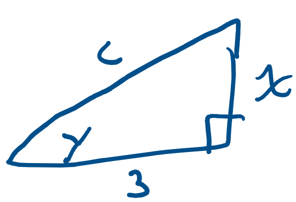
$$\tan y = \frac{x}{3}$$

$$\frac{d}{dx}(\tan y) = \frac{d}{dx}\left(\frac{x}{3}\right)$$

$$(\sec^2 y)\frac{dy}{dx} = \frac{1}{3}$$

$$\frac{dy}{dx} = \frac{1}{3\sec^2 y}$$

express y in terms of x



$\tan y = \underline{x} = \underline{opp}$

$$\tan y = \frac{x}{3} = \frac{opp}{adj}$$

$$\sec y = \frac{hyp}{adj}$$

$$hyp = c = \sqrt{x^2+3^2} = \sqrt{x^2+9}$$

$$\sec y = \left(\frac{\sqrt{x^2+9}}{3}\right)$$

$$\Rightarrow \boxed{\sec^2 y = \frac{x^2+9}{9}}$$

$$\frac{dy}{dx} = \frac{1}{3}\left(\frac{9}{x^2+9}\right)$$

$$\boxed{\frac{dy}{dx} = \frac{3}{x^2+9}}$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\boxed{\tan^2\theta + 1 = \sec^2\theta}$$

2.3

$$e = \lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x \qquad (2.1)$$

The approximate value of is $e = 2.71828182845905\ldots$ (often called the *Euler number*).

Supplied

$$e \approx \left(1 + \frac{1}{x}\right)^x \quad \Rightarrow \quad e^{1/x} \approx \left(\left(1 + \frac{1}{x}\right)^x\right)^{1/x} = 1 + \frac{1}{x} \quad \Rightarrow \quad \left(e^{1/x} - 1\right)x \approx \left(\frac{1}{x}\right)x = 1\,,$$

so letting $h = 1/x$, and noting that $h = 1/x \to 0$ if and only if $x \to \infty$, yields the useful limit:[5]

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1 \tag{2.2}$$

$$\frac{d}{dx}\left(e^x\right) = e^x$$

Proof: Using the limit definition of the derivative for $f(x) = e^x$,

$e^{x+h} = e^x \cdot e^h$

$$\frac{d}{dx}\left(e^x\right) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h\to 0} \frac{e^x\left(e^h - 1\right)}{h} = e^x \cdot \lim_{h\to 0} \frac{e^h - 1}{h} \quad \text{(since } e^x \text{ does not depend on } h\text{)}$$

$$= e^x \cdot 1 = e^x$$

QED

Memorize

$$\begin{aligned} \text{domain of } \ln x &= \text{all } x > 0 = \text{range of } e^x \\ \text{range of } \ln x &= \text{all } x = \text{domain of } e^x \end{aligned}$$

$$y = e^x \quad \text{if and only if} \quad x = \ln y \tag{2.3}$$

$$e^{\ln x} = x \quad \text{for all } x > 0 \tag{2.4}$$

$$\ln\left(e^x\right) = x \quad \text{for all } x \tag{2.5}$$

Memorize

$\ln(ab) = \ln a + \ln b$	$e^a \cdot e^b = e^{a+b}$
$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$	$\frac{e^a}{e^b} = e^{a-b}$
$\ln a^b = b \ln a$	$\left(e^a\right)^b = e^{ab}$
$\ln 1 = 0$	$e^0 = 1$

Memorize the result

To find the derivative of $y = \ln x$, use $x = e^y$:

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{d}{dy}(e^y)} = \frac{1}{e^y} = \frac{1}{x}$$

Hence:

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

How did you derive the formula for the derivative of the ln(x)?

The derivative of $\ln(x)$ can be derived from the **limit definition of the derivative** and the properties of logarithms.

Here's a step-by-step look at the derivation:

Step 1: The Limit Definition of the Derivative

The fundamental definition of a derivative is given by the formula:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

For our function, $f(x) = \ln(x)$, we substitute this into the formula:

$$\frac{d}{dx}(\ln(x)) = \lim_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h}$$

Step 2: Applying Logarithm Properties

Using the logarithm rule $\ln(a) - \ln(b) = \ln(a/b)$, we can simplify the numerator:

$$\lim_{h \to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h} = \lim_{h \to 0} \frac{1}{h} \ln\left(1 + \frac{h}{x}\right)$$

Using another logarithm rule, $c\ln(a) = \ln(a^c)$, we move the $\frac{1}{h}$ inside the logarithm:

$$\lim_{h \to 0} \ln\left(1 + \frac{h}{x}\right)^{1/h}$$

Step 3: Relating the Limit to the Number *e*

The key to solving this limit is to recognize its connection to the definition of the mathematical constant e, which is given by:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

To make our limit look like this, we perform a substitution. Let $n = \frac{x}{h}$. As h approaches 0, n approaches infinity. With this substitution, $\frac{1}{h} = \frac{n}{x}$. Our limit expression becomes:

$$\ln\left(\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{1/h}\right) = \ln\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n/x}\right)$$

We can rewrite this using the property of exponents $(a^b)^c = a^{bc}$:

$$\ln\left(\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)^{1/x}\right)$$

The limit inside the parentheses is exactly the definition of e. Therefore, the expression simplifies to:

$$\ln\left(e^{1/x}\right)$$

Step 4: Final Simplification

Using the logarithm rule $\ln(a^b) = b\ln(a)$ and the fact that $\ln(e) = 1$, we get the final result:

$$\frac{1}{x}\ln(e) = \frac{1}{x}\cdot 1 = \frac{1}{x}$$

Thus, the derivative of $\ln(x)$ is $\frac{1}{x}$.

Now that you've seen the derivation, would you like to apply this rule to a more complex function, such as finding the derivative of $y = \ln(x^3+2x)$ using the chain rule?

Memorize

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$$

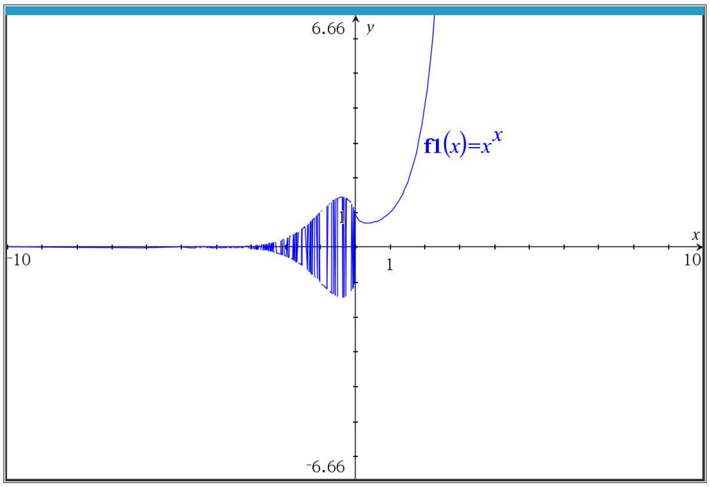
Logarithmic Differentiation

x

Logarithmic Differentiation

$$y = x^{x}$$

Find $\dfrac{dy}{dx}$



$$\ln(y) = \ln(x^{x})$$

$$\ln(y) = x \ln(x)$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln x)$$

$$\left(\frac{1}{y}\right)\frac{dy}{dx} = x \frac{d}{dx}(\ln x) + \ln x\left(\frac{dx}{dx}\right)$$

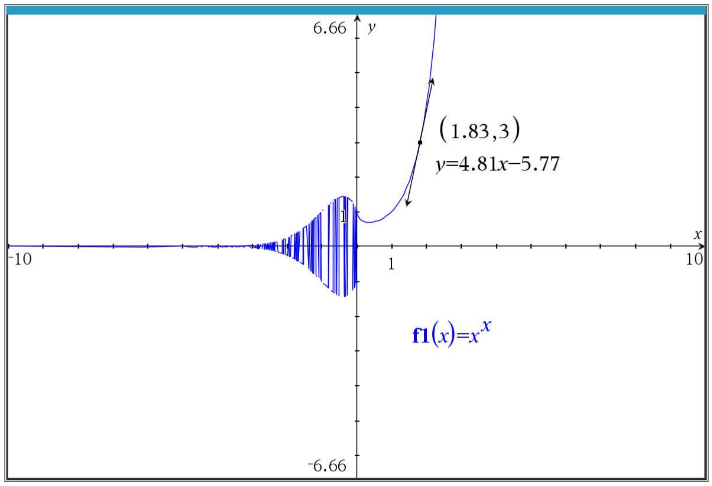
$$\left(\frac{1}{y}\right)\frac{dy}{dx} = x\left(\frac{1}{x}\right) + (\ln x)(1)$$

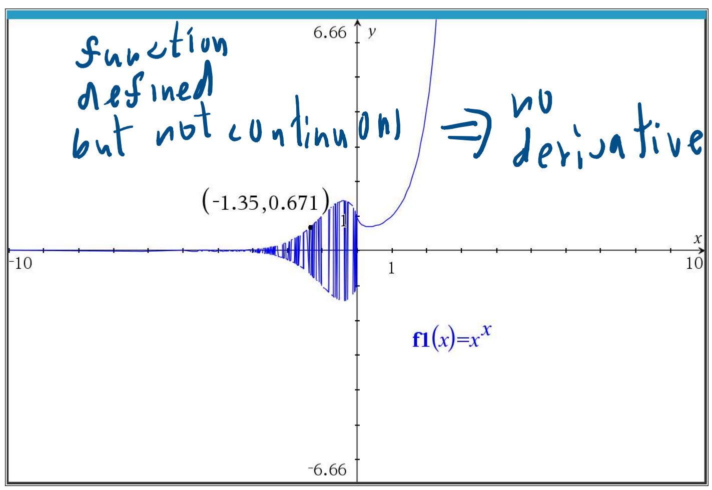
$$\left(\frac{1}{y}\right)\frac{dy}{dx} = 1 + \ln x$$

$$\frac{dy}{dx} = y(1 + \ln x)$$

$$\boxed{\frac{dy}{dx} = x^x(1+\ln x)}$$

defined for $x > 0$





function defined but not continuous $\Rightarrow$ no derivative

$$y = f(x) = \frac{(x^2+3)(2x)}{x^3+1}$$

Find $f'(x)$: hard way : quotient rule, product rule, chain rule

$$\ln(y) = \ln\left(\frac{(x^2+3)(2x)}{x^3+1}\right)$$

$$\ln(y) = \ln\left(\frac{\quad}{x^3+1}\right)$$

$$\ln(y) = \ln(x^2+3) + \ln(2x) - \ln(x^3+1)$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(\ln(x^2+3)\right) + \frac{d}{dx}\left(\ln(2x)\right) - \frac{d}{dx}\left(\ln(x^3+1)\right)$$

$$\left(\frac{1}{y}\right)\frac{dy}{dx} = \left(\frac{1}{x^2+3}\right)(2x) + \left(\frac{1}{2x}\right)(2) + \left(\frac{1}{x^3+1}\right)(3x^2)$$

$$\left(\frac{1}{y}\right)\frac{dy}{dx} = \frac{2x}{x^2+3} + \frac{1}{x} + \frac{3x^2}{x^3+1}$$

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2+3} + \frac{1}{x} + \frac{3x^2}{x^3+1}\right)$$

$$\boxed{\frac{dy}{dx} = \frac{(x^2+3)(2x)}{x^3+1}\left(\frac{2x}{x^2+3} + \frac{1}{x} + \frac{3x^2}{x^3+1}\right)}$$

Supplied, or I could ask you to derive it: half-life formula for radioactive decay

$$t_H = -\frac{\ln 2}{k} \qquad \text{and} \qquad k = -\frac{\ln 2}{t_H} \tag{2.6}$$

2.3

14. Show that $\frac{d}{dx}(\ln(x^n)) = \frac{n}{x}$ for all integers $n \geq 1$.

Let $n = 1$

show $\frac{d}{dx}(\ln x) = \frac{1}{x}$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{1}{x} = \frac{1}{x} \checkmark$$

Let $n = 2$

Show $\frac{d}{dx}\left(\ln\left(x^2\right)\right) = \frac{2}{x}$

$$\left(\frac{1}{x^2}\right)(2x)$$

$$\frac{2}{x} = \frac{2}{x} \checkmark$$

Show $\frac{d}{dx}\left(\ln\left(x^n\right)\right) = \frac{n}{x}$

$$\left(\frac{1}{x^n}\right)\left(n x^{n-1}\right)$$

$$n\left(\frac{x^{n-1}}{x^n}\right)$$

$$\frac{n}{x} = \frac{n}{x} \checkmark$$

$$\frac{d}{dx}\left(\ln\left(x^n\right)\right)$$

$$\frac{d}{dx}(n \ln x)$$

$$n \frac{d}{dx} \ln x$$

$$= \frac{n}{x}$$

Quick introduction to 2.4