09-22-25 MTH 263

2 Derivatives of Common Functions

2.1 Inverse Functions

page 50: 1, 3, 5

2.2 Trigonometric Functions and Their Inverses

page 54: 1, 4, 5, 15

Exam 1		stem & leaf		
50.21053	mean			A-0
53	median	8	6	B-1
19.73396	st. dev	7	448	C-3
17	min	6	4	D-1
86	max	5	037999	F- 14
19	count	4	03	
		3	02	
		2	289	
		1	7	

2.1: 5

A

For Exercises 1-8, show that the given function $y = f(x)$ is one-to-one over the given interval, then find the formulas for the inverse function f^{-1} and its derivative. Use Example 2.2 as a guide, including putting f^{-1} and its derivative in terms of x.

5. $f(x) = \frac{1}{x}$, for all $x > 0$

Example 2.2

Find the inverse f^{-1} of the function $f(x) = x^3$ then find the derivative of f^{-1}.

Solution: Rewrite $y = f(x) = x^3$ as $x = f(y) = y^3$, so that its inverse function $y = f^{-1}(x) = \sqrt[3]{x}$ has derivative

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{3y^2}$$, which is in terms of y, so putting it in terms of x yields

$$= \frac{1}{3\left(\sqrt[3]{x}\right)^2} = \frac{1}{3x^{2/3}}$$

which agrees with the derivative obtained by differentiating $y = \sqrt[3]{x}$ directly.

$y = f(x) = \frac{1}{x}$

$\boxed{x = f(y) = \frac{1}{y}}$

$y = \frac{1}{x}$

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}\left(x^{-1}\right)$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}\left(x^{-1}\right)$$

$$= -x^{-2} = \boxed{-\frac{1}{x^2} = \frac{dy}{dx}}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\left(\frac{1}{x}\right)^2} = -x^2$$

$y = \frac{1}{x} = f(x)$

Find $f^{-1}(x)$ and differentiate

$x = \frac{1}{y}$

solve for y

$y = \frac{1}{x}$

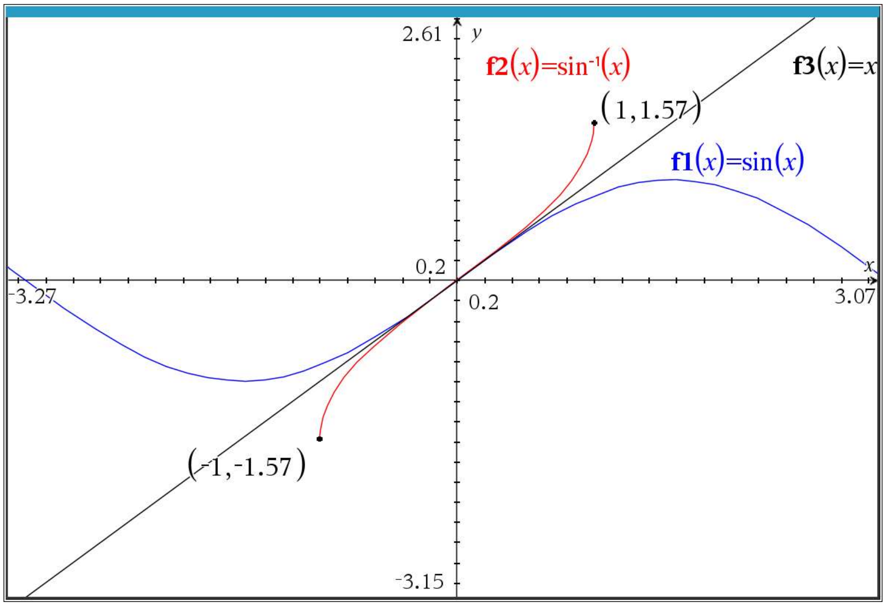
$\frac{dy}{dx} = -\frac{1}{x^2}$

$\frac{d}{dx}\left(f^{-1}(x)\right) = -\frac{1}{x^2}$

2.2

Supplied

function	$\sin^{-1} x$	$\cos^{-1} x$	$\tan^{-1} x$	$\csc^{-1} x$	$\sec^{-1} x$	$\cot^{-1} x$
domain	$[-1,1]$	$[-1,1]$	$(-\infty,\infty)$	$\lvert x\rvert \geq 1$	$\lvert x\rvert \geq 1$	$(-\infty,\infty)$
range	$[-\frac{\pi}{2},\frac{\pi}{2}]$	$[0,\pi]$	$(-\frac{\pi}{2},\frac{\pi}{2})$	$(-\frac{\pi}{2},0)\cup(0,\frac{\pi}{2})$	$(0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi)$	$(0,\pi)$



Supplied, or I will ask you to prove.

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \quad (\text{for } |x|<1) \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{|x|\sqrt{x^2-1}} \quad (\text{for } |x|>1)$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \quad (\text{for } |x|<1) \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}} \quad (\text{for } |x|>1)$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

For the derivative of $\cos^{-1}x$, recall that $y = \cos^{-1}x$ is an *angle* between 0 and π radians, defined for $-1 \le x \le 1$. Since $\cos y = x$ by the definition of y, then $\frac{dx}{dy} = -\sin y$ and

$$\sin^2 y = 1-\cos^2 y = 1-x^2 \quad \Rightarrow \quad \sin y = \pm\sqrt{1-x^2} = \sqrt{1-x^2}$$

since $0 \le y \le \pi$ (which means $\sin y$ must be nonnegative). Thus:

$$\frac{d}{dx}(\cos^{-1}x) = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{-\sin y} = -\frac{1}{\sqrt{1-x^2}} \quad \checkmark$$

$y = \cos^{-1}(x)$

$\Rightarrow \cos y = \cos(\cos^{-1}(x))$

$\Rightarrow \cos y = x$

$\frac{dx}{dy} = \frac{d}{dy}(\cos y)$

$\frac{dx}{dy} = -\sin(y)$

ay

Find $\frac{d}{dx}\left(\cos^{-1}(x)\right)$

$y = \cos^{-1}(x)$

$\Rightarrow \boxed{\cos y = x}$

implicit differentiation

Assume $y = y(x)$

$$\frac{d}{dx}(\cos y) = \frac{dx}{dx}$$

$$[-\sin(y)]\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = -\frac{1}{\sin(y)}$$

remember $\cos y = x$

$$\Rightarrow \sin(y) = \sqrt{1-\cos^2 y} = \sqrt{1-x^2}$$

$$\therefore \boxed{\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}}$$

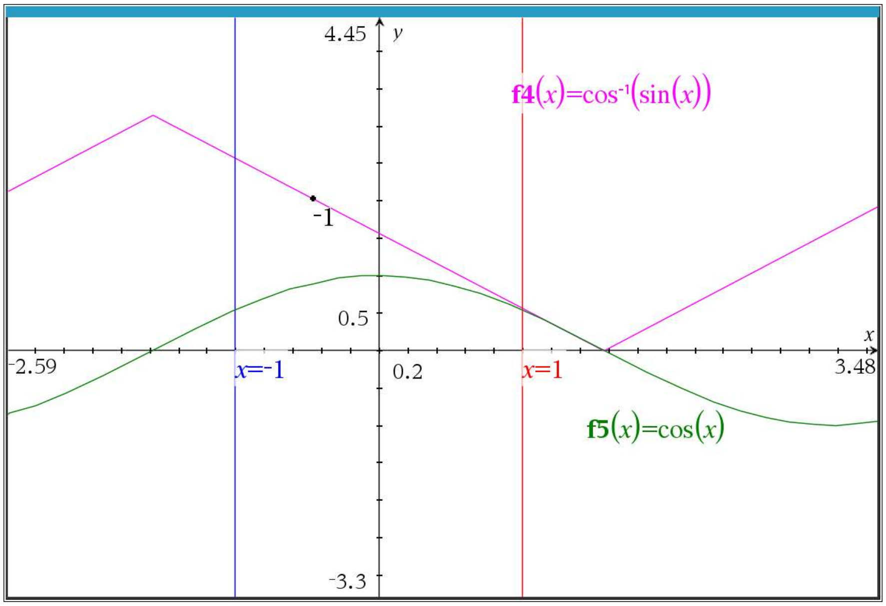
2.1

A

For Exercises 1-16, find the derivative of the given function $y = f(x)$.

8. $y = \cos^{-1}(\sin x)$ $\quad |\sin x| < 1$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \quad (\text{for } |x| < 1)$$



$$\frac{dy}{dx} = \frac{d}{dx}\left(\cos^{-1}(\sin x)\right)$$

$$= \frac{-1}{\sqrt{1-\sin^2 x}}\,\frac{d}{dx}(\sin x)$$

$$= \frac{-1}{\sqrt{1-\sin^2 x}}(\cos x)$$

$$= \frac{-1}{\sqrt{\cos^2 x}}(\cos x)$$

$$= \frac{-1}{|\cos x|}\cos x$$

$$\sqrt{x^2} = |x|$$

$$\sqrt{3^2} = \sqrt{9} = 3$$

$$\sqrt{(-3)^2} = \sqrt{9} = 3$$

on $[-1,1]$, $\cos x \geq 0$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{-1}{\cos x}\right)\cos(x) = \boxed{-1}$$

Bonus quiz 2 (group) open book, open notes, calculator OK

17. Find the derivative of $y = \sin^{-1} x + \cos^{-1} x$. Explain why no derivative formulas were needed.

Bonus quiz 2 (group) open book, open notes, calculator OK

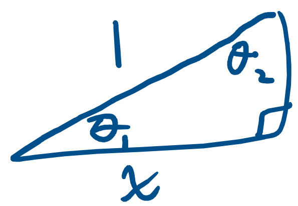
17. Find the derivative of $y = \sin^{-1}x + \cos^{-1}x$. Explain why no derivative formulas were needed.

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \quad (\text{for } |x|<1)$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \quad (\text{for } |x|<1)$$

By formulas

$$\frac{d}{dx}(\sin^{-1}x + \cos^{-1}x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$$



$\theta_1 = \cos^{-1}x$

$\theta_2 = \sin^{-1}x$

$\theta_1 + \theta_2 = \frac{\pi}{2}$

$\cos\theta_1 = x = \frac{x}{1} = \frac{\text{adj}}{\text{hyp}}$

$\sin\theta_2 = x = \frac{x}{1} = \frac{\text{opp}}{\text{hyp}}$

$\therefore \cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$

$\Rightarrow \frac{d}{dx}(\cos^{-1}x + \sin^{-1}x) = \frac{d}{dx}\left(\frac{\pi}{2}\right) = \boxed{0}$