09-17-25 MTH 263

Took exam 1

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2 Derivatives of Common Functions

2.1 Inverse Functions

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Gemini

2.5 Flash

What is the derivative of the inverse of f(x)?

The derivative of the inverse of a function $f(x)$ is given by the formula:

$$\frac{d}{dx}[f^{-1}(x)] = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

This is a direct application of the Inverse Function Theorem. It states that the derivative of the inverse function at a point x is the reciprocal of the derivative of the original function evaluated at the inverse point $f^{-1}(x)$.

Would you like me to demonstrate how to use this formula with an example, such as finding the derivative of the inverse of $f(x) = x^2$?

Yes, please.

By finding the derivative of $f(x) = x^2$ and its inverse function $f^{-1}(x) = \sqrt{x}$, we can verify the Inverse Function Theorem.

First, we find the derivatives of both functions:

- The derivative of the original function $f(x) = x^2$ is $f'(x) = 2x$.
- The derivative of the inverse function $f^{-1}(x) = \sqrt{x}$ is $(f^{-1})'(x) = \frac{1}{2\sqrt{x}}$.

Next, we apply the Inverse Function Theorem:

- We substitute the inverse function $f^{-1}(x)$ into the derivative of the original function $f'(x)$, which gives us $f'(f^{-1}(x)) = 2\sqrt{x}$.
- Then we take the reciprocal: $\frac{1}{f'(f^{-1}(x))} = \frac{1}{2\sqrt{x}}$.

As you can see, the result from the formula, $\frac{1}{2\sqrt{x}}$, matches the derivative of the inverse function calculated directly, $(f^{-1})'(x) = \frac{1}{2\sqrt{x}}$.

Would you like to try this with another function, or explore a different concept in calculus?

Definition: A function f(x) is one-to-one (1-1) if $f(x_1) = f(x_2)$

$\Rightarrow x_1 = x_2$

$f(x) = 3x + 4$

is f 1-1 ?

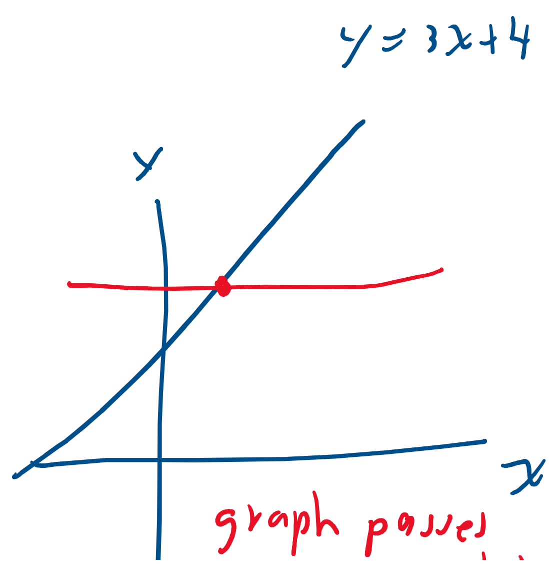
Assume $f(x_1) = f(x_2)$

$\Rightarrow 3x_1 + 4 = 3x_2 + 4$

$\Rightarrow 3x_1 = 3x_2$

$\Rightarrow x_1 = x_2$

$\therefore f$ is 1-1



$\therefore f$ is 1-1

graph passes the horizontal line test

$f(x) = x^2$

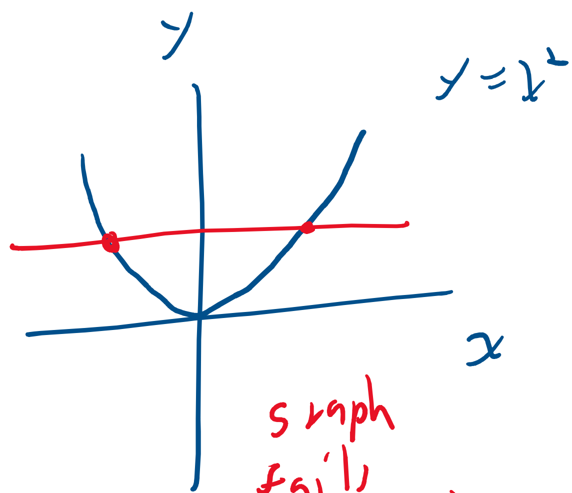
is f 1-1 ?

Assume $f(x_1) = f(x_2)$

$\Rightarrow (x_1)^2 = (x_2)^2$

$\Rightarrow x_1 = \pm x_2$

$\therefore f$ is <u>not</u> 1-1



graph fails horizontal line test

Memorize

Derivative of an Inverse Function: If $y = f(x)$ is differentiable and has an inverse function $x = f^{-1}(y)$, then f^{-1} is differentiable and its derivative is

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \qquad \text{if } \frac{dy}{dx} \neq 0.$$

Example 2.1

Find the inverse f^{-1} of the function $f(x) = x^3$ then find the derivative of f^{-1}.

Solution: The function $y = f(x) = x^3$ is one-to-one over the set of all real numbers (why?) so it has an inverse function $x = f^{-1}(y)$ defined for all real numbers, namely $x = f^{-1}(y) = \sqrt[3]{y}$.

The derivative of f^{-1} is

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{3x^2}$$

, which is in terms of x, so putting it in terms of y yields

$$= \frac{1}{3\left(\sqrt[3]{y}\right)^2} = \frac{1}{3y^{2/3}}$$

which agrees with the derivative obtained by differentiating $x = \sqrt[3]{y}$ directly. Note that this derivative is defined for all y except $y = 0$, which occurs when $x = \sqrt[3]{0} = 0$, i.e. at the point $(x, y) = (0, 0)$.

$x = \sqrt[3]{y} = y^{\frac{1}{3}}$

$\frac{dx}{dy} = \left(\frac{1}{3}\right) y^{\frac{1}{3}-1} = \left(\frac{1}{3}\right) y^{\frac{1}{3}-\frac{3}{3}} = \left(\frac{1}{3}\right) y^{-\frac{2}{3}}$

$$\frac{dx}{dy} = \left(\frac{1}{3}\right) y^{\frac{1}{3}-1} = \left(\frac{1}{3}\right) y^{\frac{1}{3}-\frac{3}{3}} = \left(\frac{1}{3}\right) y^{-\frac{2}{3}}$$
$$= \left(\frac{1}{3}\right)\left(\frac{1}{y^{2/3}}\right)$$

Example 2.2

Find the inverse f^{-1} of the function $f(x) = x^3$ then find the derivative of f^{-1}.

Solution: Rewrite $y = f(x) = x^3$ as $x = f(y) = y^3$, so that its inverse function $y = f^{-1}(x) = \sqrt[3]{x}$ has derivative

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{3y^2}$$, which is in terms of y, so putting it in terms of x yields

$$= \frac{1}{3\left(\sqrt[3]{x}\right)^2} = \frac{1}{3x^{2/3}}$$

which agrees with the derivative obtained by differentiating $y = \sqrt[3]{x}$ directly.

$$\left(f^{-1}\right)'(f(x)) = \frac{1}{f'(x)} \qquad \text{if } f'(x) \neq 0$$

Two equivalent ways to write this are:

$$\left(f^{-1}\right)'(c) = \frac{1}{f'(a)} \qquad \text{where } c = f(a) \text{ and } f'(a) \neq 0$$

and

$$\left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))} \qquad \text{if } f'(f^{-1}(x)) \neq 0$$

$y = f(x) = \sqrt{x}$

Find $\frac{d}{dx}\left(f^{-1}(x)\right)$

$f^{-1}(x) = x^2$

$\frac{d}{dx}\left(f^{-1}(x)\right) = 2x$

$\frac{d}{dx}\left(f\,(x)\right)$