09-08-25 MTH 263

1.3 The Derivatives: Infinitesimal Approach
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1.4 Derivatives of Sums, Products and Quotients
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1.3: 6

A

For Exercises 1-9, let dx be an infinitesimal and prove the given formula.

6. $\cos 2dx = 1$

$\cos(2dx) =$

$\cos^2 dx - \sin^2 dx$

$\boxed{\sin dx = dx}$

$\boxed{\cos dx = 1}$

$= 1^2 - (dx)^2$

$= 1 - 0$

$= 1$

$\cos(a+b)$

$= (\cos a)(\cos b) - (\sin a)(\sin b)$

$\cos(2a)$

$= \cos^2 a - \sin^2 a$

1.3: 13

13. Show that $\frac{d}{dx}(\cos 2x) = -2\sin 2x$. *(Hint: Use Exercises 5 and 6.)*

5. $\sin 2dx = 2dx$ **6.** $\cos 2dx = 1$

$\frac{d}{dx}(\cos 2x)$

$= \dfrac{\cos(2(x+dx)) - \cos 2x}{dx}$

$= \dfrac{\cos(2x + 2dx) - \cos 2x}{dx}$

$= (\cos(2x))(\cos(2dx)) - (\sin(2x)\sin(2dx)) - \cos 2x$

$$= \frac{(\cos(2x))(\cos(2\,dx)) - (\sin(2x)\sin(2\,dx)) - \cos 2x}{dx}$$

$$= \frac{(\cancel{\cos(2x)})(1) - (\sin 2x)(2\,dx) - \cancel{\cos 2x}}{dx}$$

$$= \frac{-2\sin 2x\, d\cancel{x}}{d\cancel{x}}$$

$$\boxed{\frac{d}{dx}(\cos 2x) = -2\sin 2x}$$

Prove

5. $\sin 2dx = 2dx$

$$\sin(2\,dx) = 2\sin dx \cos dx$$

$$= 2(dx)(1)$$

$$= 2\,dx$$

1.4
Memorize

Rules for Derivatives: Suppose that f and g are differentiable functions of x. Then:

Sum Rule: $\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$

Difference Rule: $\frac{d}{dx}(f-g) = \frac{df}{dx} - \frac{dg}{dx}$

Constant Multiple Rule: $\frac{d}{dx}(cf) = c \cdot \frac{df}{dx}$ for any constant c

Product Rule: $\frac{d}{dx}(f \cdot g) = f \cdot \frac{dg}{dx} + g \cdot \frac{df}{dx}$

Quotient Rule: $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g \cdot \frac{df}{dx} - f \cdot \frac{dg}{dx}}{g^2}$

Quotient Rule: $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g \cdot \frac{df}{dx} - f \cdot \frac{dg}{dx}}{g^2}$

Product rule: standard derivation.

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right), \text{ given } f', g' \text{ exist}$$

$$= \lim_{h\to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)\,g(x+h) + \overbrace{\left(f(x)\,g(x+h) - f(x)\,g(x+h)\right)}^{=0} - f(x)\,g(x)}{h}$$

$$= \lim_{h\to 0} \frac{\left(f(x+h)\,g(x+h) - f(x)\,g(x+h)\right) + \left(f(x)\,g(x+h) - f(x)\,g(x)\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left[f(x+h) - f(x)\right]g(x+h) + f(x)\left[g(x+h) - g(x)\right]}{h}$$

$$= \lim_{h\to 0} \frac{\left[f(x+h) - f(x)\right]g(x+h)}{h} + \lim_{h\to 0} \frac{f(x)\left[g(x+h) - g(x)\right]}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h\to 0} g(x+h) + f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) \cdot \lim_{h\to 0} g(x+h) + f(x)\,g'(x)$$

$= f(x) \cdot \lim_{h\to 0} g(x+h) + f(x)\, g'(x)$

we will prove if a function is differentiable,
then it is continuous $\Rightarrow \lim_{h\to 0} g(x+h) = g(x+0)$
$= g(x)$

$= f(x)\, g(x) + f(x)\, g'(x)$

Memorize (or be able to derive)

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

We can use the first two to derive the other four.

Memorize

For $n \geq 1$ differentiable functions $f_1, \ldots, f_n$ and constants $c_1, \ldots, c_n$:

$$\frac{d}{dx}(c_1 f_1 + \cdots + c_n f_n) = c_1 \frac{df_1}{dx} + \cdots + c_n \frac{df_n}{dx}$$

Memorize

Power Rule: $\frac{d}{dx}(x^n) = n\,x^{n-1}$ for any integer n

$$\frac{d}{dx}(x^0) = \frac{d}{dx}(1) = 0$$

rule: $\frac{d}{dx}(x^0) = 0 \cdot x^{0-1} = 0 \cdot x^{-1} = 0 \cdot \frac{1}{x} = 0$ if $x \neq 0$

$0^0 = ?$

0^0	undef

$$\frac{d}{dx}(x^1) = \frac{d}{dx}(x) = 1$$

rule $\frac{d}{dx}(x^1) = 1 \cdot x^{1-1} = 1 \cdot x^0 = 1 \cdot 1 = 1$

$$\frac{d}{dx}(x^2) = \frac{d}{dx}(x \cdot x) = x\frac{dx}{dx} + x\frac{dx}{dx}$$

product rule

$$= x(1) + x(1)$$

$$= 2x$$

rule $\frac{d}{dx}(x^2) = 2x^{2-1} = 2x^1 = 2x$

Memorize

In general, the derivative of a polynomial of degree $n \geq 0$ is given by:

For any constants $a_0, \ldots, a_n$ with $n \geq 0$:

$$\frac{d}{dx}\left(a_n x^n + a_{n-1}x^{n-1} \cdots + a_2 x^2 + a_1 x + a_0\right) = n a_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2 x + a_1$$

$$\frac{d}{dx}\left(5x^3 - 2x^2 + 7x + 4\right)$$

$$= \frac{d}{dx}(5x^3) + \frac{d}{dx}(-2x^2) + \frac{d}{dx}(7x) + \frac{d}{dx}(4)$$

$$= 5\frac{d}{dx}(x^3) - 2\frac{d}{dx}(x^2) + 7\frac{d}{dx}(x) + 0$$

$$= 5(3x^2) - 2(2x) + 7(1)$$

$= 5(3x^2) - 2(2x) + 7(1)$

$= \boxed{15x^2 - 4x + 7}$

Supplied

Principle of Mathematical Induction

A statement $P(n)$ about integers $n \geq k$ is true for all $n \geq k$ if:

1. $P(k)$ is true.
2. If $P(n)$ is true for some integer $n \geq k$ then $P(n+1)$ is true.

Prove $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + (n-1) + n$

basis step Let $n = 1$

Prove $\sum_{k=1}^{1} k = \frac{1(1+1)}{2}$

$1 \;\Big|\; \frac{1(2)}{2}$

$1 = 1$ ✓

Induction hypothesis

Assume $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ for any fixed $n \geq 1$

$n \quad \sum^{n+1} \quad \frac{(n+1)(n+1+1)}{}$

$k=1$

Prove $\displaystyle\sum_{k=1}^{n+1} k = \frac{(n+1)(n+1+1)}{2}$

i.e. prove $\displaystyle\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$

$$\sum_{k=1}^{n+1} k = \left(\sum_{k=1}^{n} k\right) + n+1$$

$$= \frac{n(n+1)}{2} + (n+1) \quad \text{Ind hyp.}$$

$$= (n+1)\left(\frac{n}{2} + 1\right)$$

$$= (n+1)\left(\frac{n}{2} + \frac{2}{2}\right)$$

$$= \frac{(n+1)(n+2)}{2} \quad \checkmark$$

Prove $2^n \geq n+1$

basis step Let $n=1$

$$2^1 \overset{?}{\geq} 1+1$$

$$2 \geq 2 \quad \checkmark$$

Ind hyp Assume $2^n \geq n+1$ for any fixed $n \geq 1$

Prove $n+1$

Ind hyp ... $\geqslant n+1$ ...

Prove $2^{n+1} \geqslant n+2$

Ind hyp $2^n \geqslant n+1$

$$\Rightarrow 2 \cdot 2^n \geqslant 2(n+1)$$

$$\Rightarrow 2^{n+1} \geqslant 2n+2 = n+(n+2) \geqslant n+2$$

Geometric check