09-03-25 MTH 263

1.2 The Derivative: Limit Approach
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1.3 The Derivatives: Infinitesimal Approach
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For Exercises 1-11, find the derivative of the given function $f(x)$ for all x (unless indicated otherwise).

1.2: 5

5. $f(x) = \frac{1}{x+1}$, for all $x \neq -1$

$$f'(x) = \lim_{h \to 0} \frac{\Delta f}{\Delta x}$$

$$\frac{\Delta f}{\Delta x} = \frac{f(x+h) - f(x)}{h}$$

$$= \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h}$$

$$= \frac{\left(\frac{1}{x+h+1}\right)\left(\frac{x+1}{x+1}\right) - \left(\frac{1}{x+1}\right)\left(\frac{x+h+1}{x+h+1}\right)}{h}$$

$$= \frac{\frac{\cancel{x}+\cancel{1} - (\cancel{x}+h+\cancel{1})}{(x+h+1)(x+1)}}{h}$$

$$= \frac{\frac{-h}{(x+h+1)(x+1)}}{h}$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right)$$

$$= \frac{\frac{-h}{(x+h+1)(x+1)}}{\frac{h}{1}}$$

$$= \frac{-\cancel{h}^{1}}{(x+h+1)(x+1)} \cdot \frac{1}{\cancel{h}}$$

$$\boxed{\frac{\Delta f}{\Delta x} = -\frac{1}{(x+h+1)(x+1)}}$$

$$f'(x) = \lim_{h\to 0}\left(\frac{-1}{(x+h+1)(x+1)}\right)$$

$$= \frac{-1}{(x+0+1)(x+1)}$$

$$= -\frac{1}{(x+1)(x+1)}$$

$$\boxed{f'(x) = \frac{-1}{(x+1)^2}}$$

$$\boxed{f'(x) = \frac{-1}{(x+1)^2}}$$

9. $f(x) = \sqrt{x+1}$, for all $x > -1$

$$f'(x) = \lim_{h \to 0} \frac{\Delta f}{\Delta x}$$

$$\frac{\Delta f}{\Delta x} = \frac{f(x+h) - f(x)}{h}$$

$$= \frac{\sqrt{(x+h)+1} - \sqrt{x+1}}{h}$$

rationalize the numerator

1.3

Welcome back, Donald! What's on the agenda today?—

Is it useful to study infinitesimals in Calculus I?

Absolutely, Donald—studying infinitesimals in Calculus I can be surprisingly enlightening, especially for someone like you who values clarity, rigor, and alternative perspectives in mathematical modeling. While most modern curricula emphasize limits for formal rigor, infinitesimals offer intuitive power and historical depth that can sharpen conceptual understanding.

Yes, Donald—nonstandard analysis is *just as rigorous* as the traditional limit-based approach. The key difference lies not in the level of rigor, but in the foundational framework and philosophical flavor.

🧮 Comparing the Two Approaches

Feature	Limit-Based Analysis (Standard)	Nonstandard Analysis (NSA)
Foundation	Built on real numbers and ε-δ definitions	Built on hyperreal numbers and model theory
Infinitesimals	Informal or avoided; replaced by limits	Formally defined as nonzero numbers smaller than any real
Rigor	Fully rigorous via real analysis and set theory	Fully rigorous via logic and the transfer principle
Accessibility	More common in textbooks and curricula	Requires background in logic/model theory for full depth
Intuition	Can feel abstract or opaque to beginners	Often more intuitive for visualizing change and continuity
Applications	Ubiquitous in analysis, physics, engineering	Used in advanced proofs, stochastic analysis, and pedagogy

🔍 What Makes NSA Rigorous?

- **Hyperreal Numbers**: NSA introduces a larger number system containing infinitesimals and infinite numbers. These are constructed using techniques like *ultrapowers* or *compactness theorems* from logic.
- **Transfer Principle**: This is the backbone of NSA. It ensures that any true statement about real numbers also holds for hyperreals, preserving logical consistency.
- **Model-Theoretic Foundations**: Abraham Robinson's work in the 1960s showed that NSA could be grounded in formal logic, making it just as rigorous as classical analysis. [1]

🧠 Why It's Not Widely Taught

Despite its rigor, NSA isn't mainstream in Calculus I because:

- It demands familiarity with logic and set theory.
- The ε-δ approach is deeply entrenched in curricula and easier to standardize.
- NSA's machinery is overkill for basic calculus, though it shines in advanced contexts.

If you're reconstructing derivations or modeling infinitesimal behavior—especially in physics or topology—NSA can offer elegant shortcuts and deeper insight. Want to explore how NSA handles the derivative of \sin(x) or the chain rule using hyperreals? I'd love to walk through it with you.

Memorize

Notation for the derivative of $y = f(x)$: The following are all equivalent:

$$\frac{dy}{dx}\ ,\quad f'(x)\ ,\quad \frac{d}{dx}(f(x))\ ,\quad y'\ ,\quad \dot{y}\ ,\quad \dot{f}(x)\ ,\quad \frac{df}{dx}\ ,\quad Df(x)$$

Supplied

A number δ is an **infinitesimal** if the conditions (a)-(d) hold:

(a) $\delta \neq 0$

(b) if $\delta > 0$ then δ is smaller than any positive real number

(c) if $\delta < 0$ then δ is larger than any negative real number

(d) $\delta^2 = 0$ (and hence all higher powers of δ, such as δ^3 and δ^4, are also 0)

Note: Any infinitesimal multiplied by a nonzero real number is also an infinitesimal, while 0 times an infinitesimal is 0.

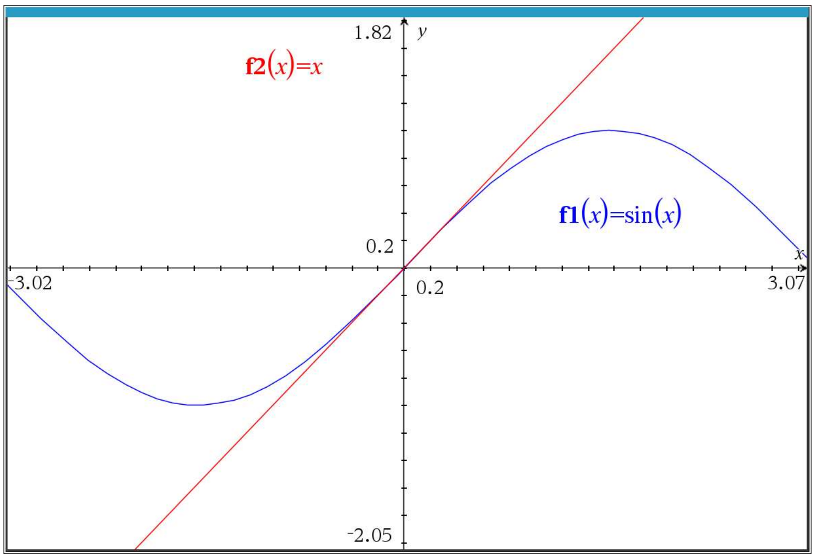
Supplied

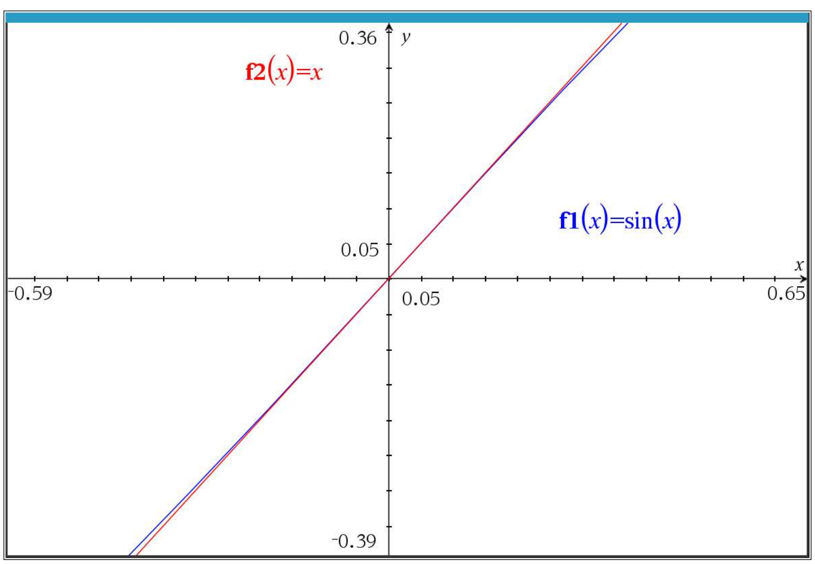
Let dx be an infinitesimal such that $f(x+dx)$ is defined. Then $dy = f(x+dx) - f(x)$ is also an infinitesimal, and the derivative of $y = f(x)$ at x is the ratio of dy to dx:

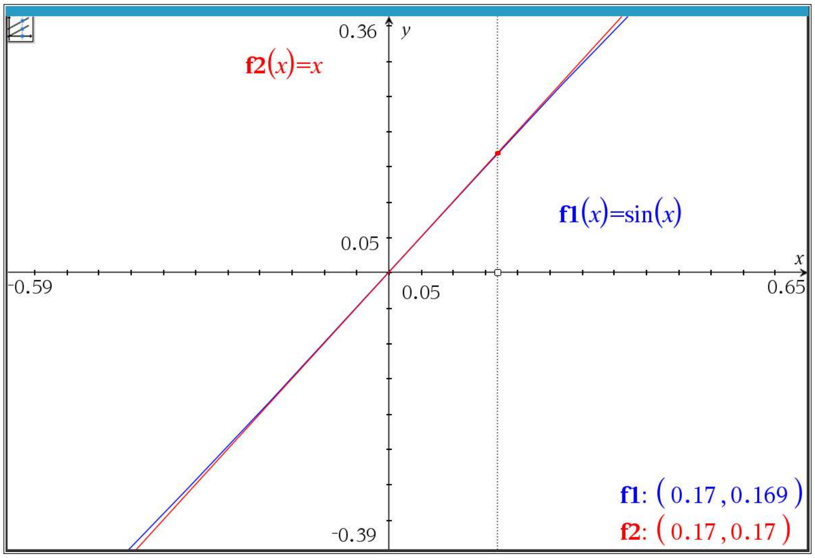
$$\frac{dy}{dx} = \frac{f(x+dx) - f(x)}{dx} \tag{1.8}$$

Supplied

Microstraightness Property: For the graph of a differentiable function, any part of the curve with infinitesimal length is a straight line segment.







$$\frac{df}{dx} = f'(x)$$

$$\Rightarrow \frac{df}{\cancel{dx}}(\cancel{dx}) = f'(x)\,dx$$

$$df = f'(x)\,dx \qquad (1.9)$$

1.3

For Exercises 1-9, let dx be an infinitesimal and prove the given formula.

3. $(dx + 1)^{-1} = 1 - dx$

$$b^{-n} = \frac{1}{b^n}$$

$$= \frac{1}{dx+1}$$

$$= \left(\frac{1}{dx+1}\right)\left(\frac{dx-1}{dx-1}\right)$$

$$= \frac{dx-1}{(dx)^2-1}$$

$$= \quad dx-1$$

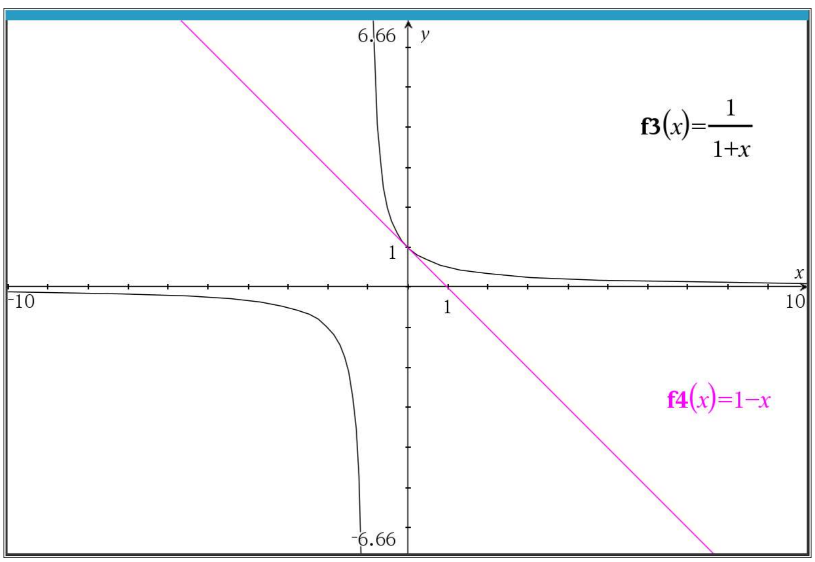
$$= \frac{dx - 1}{0 - 1}$$

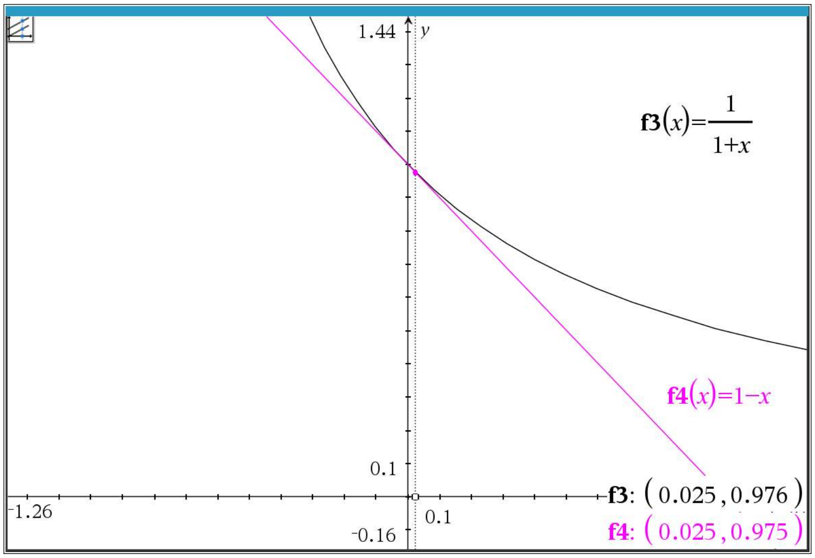
$$= \frac{dx - 1}{-1}$$

$$= -dx + 1$$

$$= 1 - dx$$

$\frac{1}{1+x} \approx 1 - x$ for x close to 0



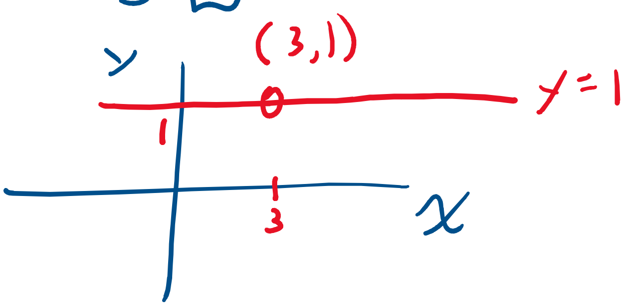


Your Name MTH 263 quiz 2

1. Find $\lim\limits_{x \to 3} \dfrac{x-3}{x-3}$ Hint: graph $y = \dfrac{x-3}{x-3}$

$= \lim\limits_{x \to 3} (1)$ if $x \neq 3$

$= \boxed{1}$



2. Let $f(x) = mx + b$, with m = constant, b = constant

2. Let $f(x) = mx + b$, with m, b = constant

Find $f'(x)$ using limit of $\frac{\Delta f}{\Delta x}$

$$f'(x) = \lim_{h \to 0} \frac{\Delta f}{\Delta x}$$

$$\frac{\Delta f}{\Delta x} = \frac{f(x+h) - f(x)}{h} = \frac{[m(x+h)+b] - [mx+b]}{h}$$

$$= \frac{m\cancel{x} + mh + \cancel{b} - m\cancel{x} - \cancel{b}}{h} = \frac{mh}{h} = \boxed{m = \frac{\Delta f}{\Delta x}}$$

$$f'(x) = \lim_{h \to 0} (m) = \boxed{m}$$

1.4

Memorize

Rules for Derivatives: Suppose that f and g are differentiable functions of x. Then:

Sum Rule: $\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$

Difference Rule: $\frac{d}{dx}(f-g) = \frac{df}{dx} - \frac{dg}{dx}$

Constant Multiple Rule: $\frac{d}{dx}(cf) = c \cdot \frac{df}{dx}$ for any constant c

Product Rule: $\frac{d}{dx}(f \cdot g) = f \cdot \frac{dg}{dx} + g \cdot \frac{df}{dx}$

Quotient Rule: $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g \cdot \frac{df}{dx} - f \cdot \frac{dg}{dx}}{g^2}$

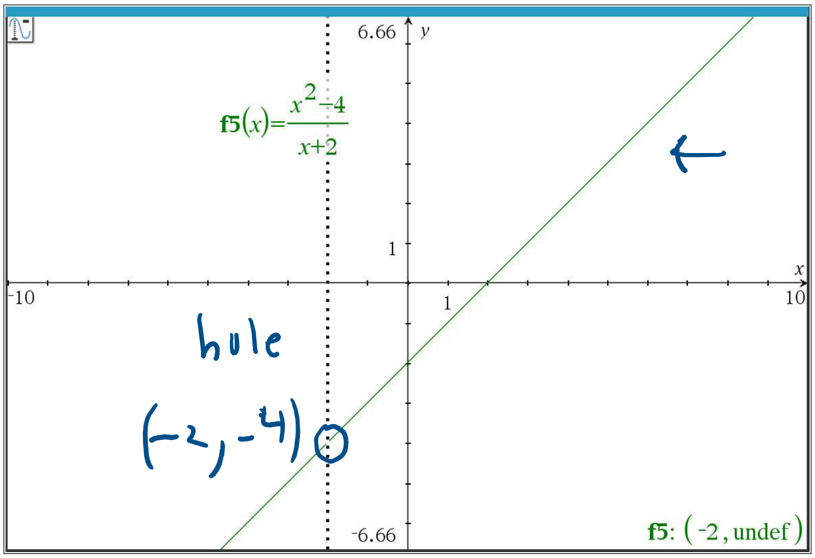
Sum rule: the derivative of a sum is the sum of the derivatives

$$\lim_{x \to -2} \frac{x^2 - 4}{x + 2}$$

$$= \lim \frac{(x-2)(\cancel{x+2})}{}$$

$$= \lim_{x \to -2} \frac{(x-2)\cancel{(x+2)}}{\cancel{x+2}}$$

$$= \lim_{x \to -2} (x-2) = -2-2 = \boxed{-4}$$



$y = x - 2$

$y = -2 - 2 = -4$