1 The Derivative

1.1 Introduction

page 16: 2, 5

1.2 The Derivative: Limit Approach

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Quiz 1

Let $f(x) = 3x + 5$

Write and simplify the difference quotient

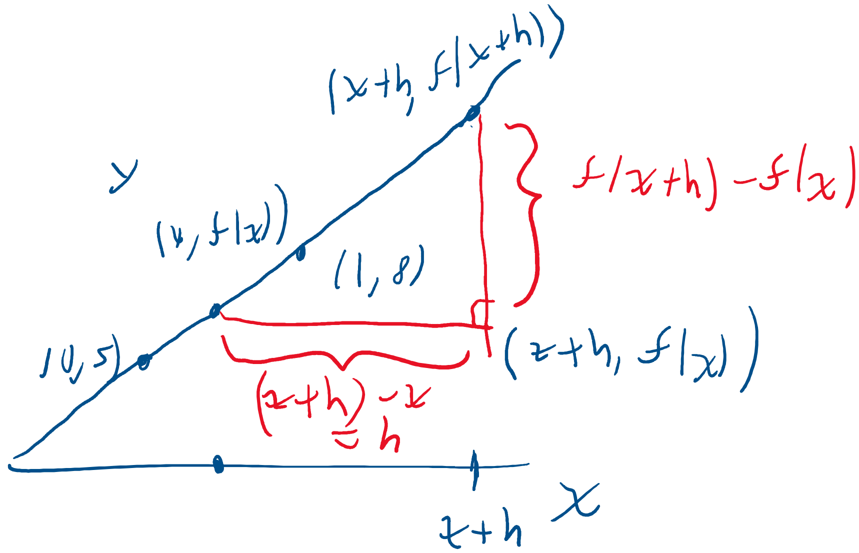
$\frac{\Delta f}{\Delta x} = \frac{f(x+h)-f(x)}{h},\ h \neq 0$

$$= \frac{[3(x+h)+5] - (3x+5)}{h}$$

$$= \frac{\cancel{3x} + 3h + \cancel{5} - \cancel{3x} - \cancel{5}}{h}$$

$$= \frac{3\cancel{h}}{\cancel{h}}$$

$$\boxed{\frac{\Delta f}{\Delta x} = 3}$$



slope of secant line

$= \frac{\Delta f}{\Delta x}$

1.2

Memorize

The **derivative** of a real-valued function $f(x)$, denoted by $f'(x)$, is

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{1.3}$$

for x in the domain of f, provided that the limit exists.[11]

$$f(x) = 3x + 5$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} (3) = 3$$

$$\boxed{\therefore f'(x) = 3}$$

Memorize

The derivative of any constant function is 0.

$f(x) = x$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(\cancel{x} + \Delta x) - \cancel{x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = \lim_{\Delta x \to 0} 1 = 1$$

Memorize

The derivative of any linear function is the slope of the line itself:
If $f(x) = mx + b$ then $f'(x) = m$ for all x.

Memorize

For a real number a and a real-valued function $f(x)$, say that the *limit* of $f(x)$ as x approaches a equals the number L, written as

$$\lim_{x \to a} f(x) = L ,$$

if $f(x)$ approaches L as x approaches a.
Equivalently, this means that $f(x)$ can be made as close as you want to L by choosing x close enough to a. Note that x can approach a from any direction.

Memorize

Rules for Limits: Suppose that a is a real number and that $f(x)$ and $g(x)$ are real-valued functions such that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist. Then:

(a) $\lim\limits_{x \to a} (f(x) + g(x)) = \left(\lim\limits_{x \to a} f(x)\right) + \left(\lim\limits_{x \to a} g(x)\right)$

(b) $\lim\limits_{x \to a} (f(x) - g(x)) = \left(\lim\limits_{x \to a} f(x)\right) - \left(\lim\limits_{x \to a} g(x)\right)$

(c) $\lim\limits_{x \to a} (k \cdot f(x)) = k \cdot \left(\lim\limits_{x \to a} f(x)\right)$ for any constant k

(d) $\lim\limits_{x \to a} (f(x) \cdot g(x)) = \left(\lim\limits_{x \to a} f(x)\right) \cdot \left(\lim\limits_{x \to a} g(x)\right)$

(e) $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$, if $\lim\limits_{x \to a} g(x) \neq 0$

(a) the limit of a sum is the sum of the limits

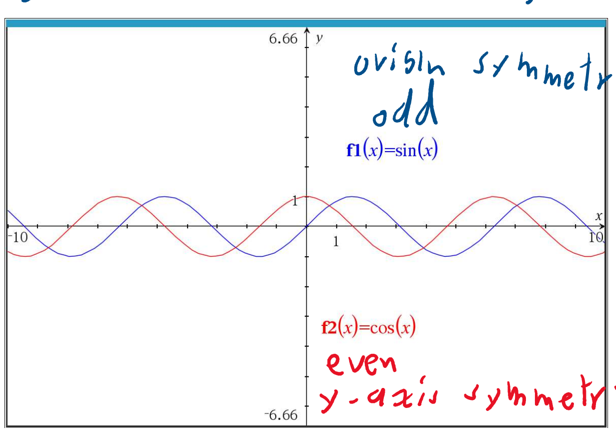
Supplied if needed.

The derivative of an even function is an odd function.
The derivative of an odd function is an even function.

def: f is odd if $f(-x) = -f(x)$, $\forall x \in$ domain of f

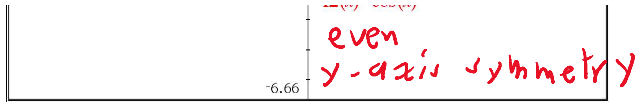
$\forall$ = for any

def f is even if $f(-x) = f(x)$, $\forall x \in$ domain of f



origin symmetry
odd

even
y-axis symmetry



Logic

Proposition $p \Longrightarrow q$

implies

p and q are statements

converse $q \Longrightarrow p$

Inverse $\sim p \Longrightarrow \sim q$

not

conrtrapositive $\sim q \Longrightarrow \sim p$

$\Longleftrightarrow$

iff

if and only if

1.2

For Exercises 1-11, find the derivative of the given function $f(x)$ for all x (unless indicated otherwise).

4. $f(x) = 2x^2 - 3x + 1$

[rules of differentiation
$f'(x) = 4x - 3$]

$$f'(x) = \lim_{h \to 0} \frac{\Delta f}{\Delta x}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left[2(x+h)^2 - 3(x+h) + 1\right] - \left[2x^2 - 3x + 1\right]}{h}$$

$$= \lim_{h \to 0} \frac{\left[\cancel{2x^2} + 4xh + 2h^2 - \cancel{3x} - 3h \cancel{+1} - \cancel{2x^2} + \cancel{3x} \cancel{-1}\right]}{h}$$

$$= \lim_{h \to 0} \frac{4xh + 2h^2 - 3h}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(4x + 2h - 3)}{\cancel{h}}$$

$$= \lim_{h \to 0} (4x + 2h - 3)$$

$$= 4x + (2)(0) - 3$$

$$\boxed{f'(x) = 4x - 3}$$

Alternative presentation

$$\frac{\Delta f}{\Delta x} = \frac{f(x+h) - f(x)}{h} = \cdots = 4x + 2h - 3$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} (4x + 2h - 3) = \boxed{4x-3}$$

8. $f(x) = \sqrt{x}$, for all $x > 0$ *(Hint: Rationalize the numerator in the definition of the derivative.)*

$$\frac{\Delta f}{\Delta x} = \frac{f(x+h) - f(x)}{h}$$

$$= \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$a^2 - b^2$

$= (a+b)(a-b)$

$$= \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \right) \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right)$$

$$= \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$\frac{\Delta f}{\Delta x} = \frac{\cancel{h}\ 1}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$f'(x) = \lim_{h \to 0} \frac{\Delta f}{\Delta x}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x+0} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}}$$

$$\boxed{f'(x) = \frac{1}{2\sqrt{x}}}$$