04-16-26 MTH 261

Section 3.6: Area, Volume, and Average Value (Area only)
page 213: 1, 3, 5, 7, 11

Section 3.7: Applications to Business
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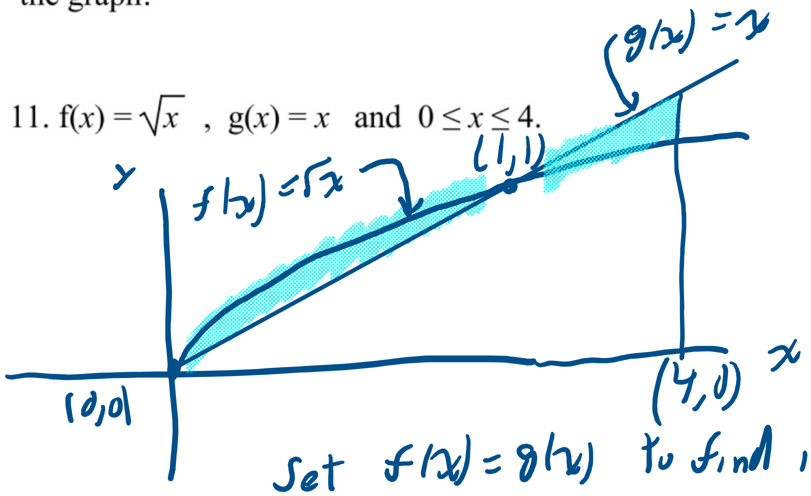
3.6: 11

In problems 6 – 15, find the **area** between the graphs of f and g for x in the given interval. Remember to draw the graph!

11. $f(x) = \sqrt{x}$, $g(x) = x$ and $0 \le x \le 4$.



Set $f(x) = g(x)$ to find intersection points

$x = \sqrt{x}$

$x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0, 1$

$\Rightarrow y = 0, 1$

$f\left(\frac{1}{4}\right) = \sqrt{\frac{1}{4}} = \frac{1}{2}$

$g\left(\frac{1}{4}\right) = \frac{1}{4}$

$\frac{1}{4} < \frac{1}{2}$

$\therefore f \geq g$ on $[0, 1]$

Google Gemini

Please give me one hint to help me solve this problem. Don't give the entire solution.

Analysis

To find the area between these two curves, the most important first step is to determine which function is "on top" throughout the interval.

The Hint

While the interval is given as $[0, 4]$, these two graphs actually **intersect** at a point between 0 and 4.

Because they cross each other, the "upper" function and "lower" function swap places at that intersection point. To find the total area, you will need to:

1. **Find the intersection point** by setting $f(x) = g(x)$.
2. **Split the integral** into two separate parts: one from 0 to the intersection point, and another from that point to 4.

This ensures that you are always subtracting the bottom function from the top function, which keeps your area value positive for both sections.

$$\text{Area} = \int_0^1 (f(x) - g(x))\,dx + \int_1^4 (g(x) - f(x))\,dx$$

$$\int_0^1 (\sqrt{x} - x)\,dx + \int_1^4 (x - \sqrt{x})\,dx$$

$$\int_0^1 (x^{\frac{1}{2}} - x)\,dx + \int_1^4 (x - x^{\frac{1}{2}})\,dx$$

$$= \left[\frac{2x^{\frac{3}{2}}}{3} - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - \frac{2x^{\frac{3}{2}}}{3}\right]_1^4$$

$$= \left[\frac{2(1^{\frac{3}{2}})}{3} - \frac{1^2}{2}\right] - [0] + \left[\frac{4^2}{2} - \frac{2(4^{\frac{3}{2}})}{3}\right]$$

$$- \left[\frac{1^2}{2} - \frac{2(1^{\frac{3}{2}})}{3}\right]$$

$$= \frac{2}{3} - \frac{1}{2} + 8 - \frac{16}{3} - \frac{1}{2} + \frac{2}{3}$$

$$= -\frac{12}{3} - 1 + 8$$

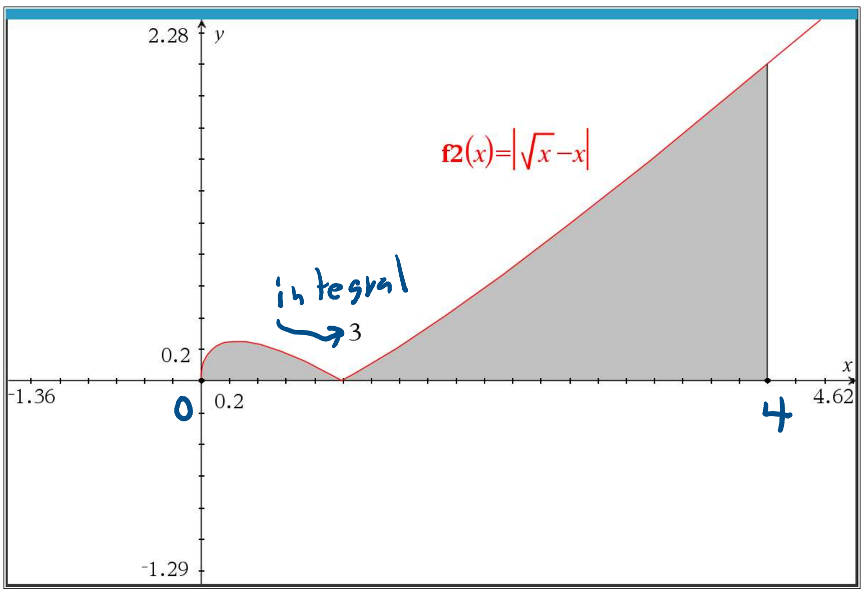
$$= -4 - 1 + 8$$

$= -4 - 1 + 8$

$= \boxed{3}$

$\int_0^1 (\sqrt{x} - x)\,dx + \int_1^4 (x - \sqrt{x})\,dx$

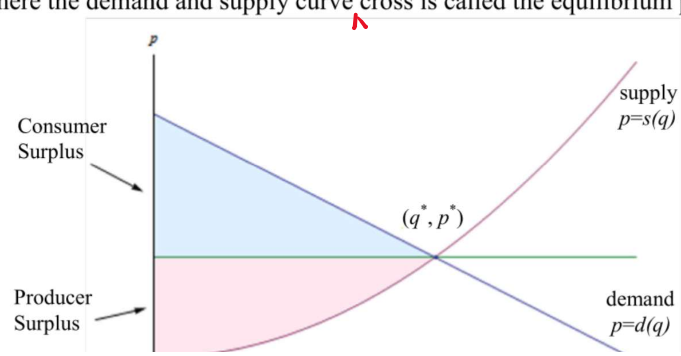
$= \int_0^4 |\sqrt{x} - x|\,dx$



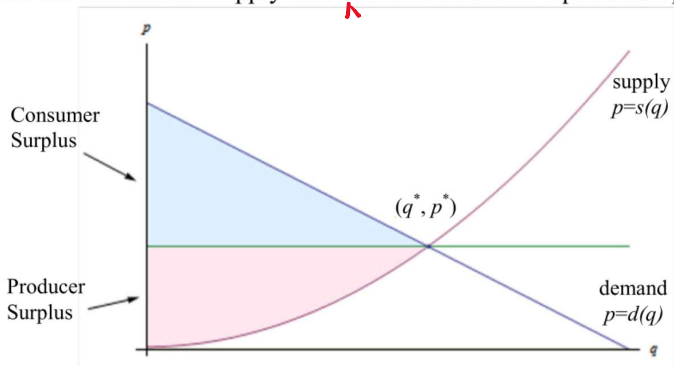
3.7

Supplied

The point where the demand and supply curves cross is called the equilibrium point (q*, p*).



The point where the demand and supply curve cross is called the equilibrium point (q*, p*).



Short answer: Yes — it should be **"curves."**

The sentence is referring to **two** distinct objects: the demand curve and the supply curve. Because both are being named, standard English requires the plural:

- "the demand and supply **curves** cross"

Using the singular "curve" would imply there is only one curve that is both demand and supply, which isn't what the sentence means.

A quick way to explain it to students

When two nouns share an article ("the demand and supply ___"), the missing noun is implicitly repeated:

- "the demand ___ and [the] supply ___"

So the plural is the natural choice.

supplied

Consumer and Producer Surplus

Given a demand function p = d(q) and a supply function p = s(q), and the equilibrium point (q*, p*)

The **consumer surplus** = $\int_0^{q^*} d(q)\,dq - p^* q^*$

The **producer surplus** = $p^* q^* - \int_0^{q^*} s(q)\,dq$

The sum of the consumer surplus and producer surplus is the **total gains from trade**.

Supplied

Compound Interest Formulas

Let P = the principal (initial investment), r = the annual interest rate expressed as a decimal, and let A(t) be the amount in the account at the end of t years.

Compounding *n* times per year: $A(t) = P\left(1+\frac{r}{n}\right)^{nt}$

Compounded continuously: $A(t) = Pe^{rt}$

If you're using this formula to find what an account will be worth in the future, t > 0 and A(t) is called the **future value**.

If you're using the formula to find what you need to deposit today to have a certain value P sometime in the future, t < 0 and A(t) is called the **present value.**

Supplied

Continuous Income Stream

Suppose money can earn interest at an annual interest rate of r, compounded continuously. Let F(t) be a continuous income function (in dollars per year) that applies between year 0 and year T.

Then the present value of that income stream is given by $PV = \int_0^T F(t)e^{-rt}\,dt$.

The future value can be computed by the ordinary compound interest formula $FV = PVe^{rt}$

Example 3

You have an opportunity to buy a business that will earn \$75,000 per year continuously over the next eight years. Money can earn 2.8% per year, compounded continuously. Is this business worth its purchase price of \$630,000?

First, please note that we still have to make some simplifying assumptions. We have to assume that the interest rates are going to remain constant for that entire eight years. We also have to assume that the \$75,000 per year is coming in continuously, like a faucet dripping dollars into the business. Neither of these assumptions might be accurate.

But moving on:
The present value of the \$630,000 is, well, \$630,000. This is one investment, where we put our \$630,000 in the bank and let it sit there.

To find the present value of the business, we think of it as an income stream. The function F(t) in this case is a constant \$75,000 dollars per year, so F(t) = 75,000 . The interest rate is 2.8% and the term we're interested in is 8 years, so r = .028, and T = 8:

$$PV = \int_0^8 75000e^{-.028t}\,dt \cong 672{,}511.66$$

The present value of the business is about \$672,500, which is more than the \$630,000 asking price – this is a good deal.

In class, I misinterpreted this problem. I thought that the choice was between buying the business or investing the money in a savings account. If the question is simply: is the business worth its purchase price, then I agree with the textbook analysis.

However, what would happen if we put the \$630,000 in a savings account that earned 2.8% annual interest, compounded continuously?

Compounded continuously: $A(t) = Pe^{rt}$

$$A(8) = \$630{,}000\, e^{(0.028)(8)}$$

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630000*exp(0.028*8)
                788174.7422
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788174.7422-630000=158174.7422
After 8 years, we have earned \$158,175 above our initial investment.

If we buy the business, we earn $75,000 per year, plus 2.8% interest compounded continuously. This calculation becomes messy, because we must add $75,000 to our earnings at the end of the first year, and then calculate the continuously compounding interest for year 2. Then, we repeat this process with a new principal for year 3, 4, 5, 6, 7, and 8. We must also take into account that we not only have the cash earned, but we still own the business.