

Section 2.11: Implicit Differentiation and Related Rates

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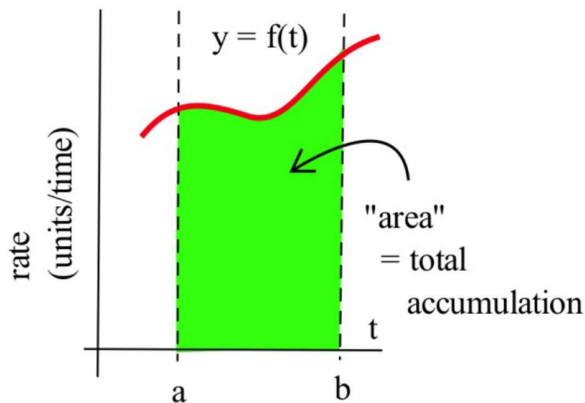
Chapter 3: The Integral

PreCalculus Idea – The Area of a Rectangle

Section 3.1: The Definite Integral

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3.1



$$t = \text{time}$$

$$y = \text{rate} \left(\frac{\text{units}}{\text{time}} \right)$$

Problem: how do we calculate the area under the curve, above the t -axis, and between $t = a$ and $t = b$?

Memorize

Riemann sum: A Riemann sum for a function $f(x)$ over an interval $[a, b]$ is a sum of areas of rectangles that approximates the area under the curve. Start by dividing the interval $[a, b]$ into n subintervals; each subinterval will be the base of one rectangle. We usually make all the rectangles the same width Δx . The height of each rectangle comes from the function evaluated at some point in its sub interval. Then the Riemann sum is:

$$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x$$

Sigma Notation: The upper-case Greek letter Sigma Σ is used to stand for Sum. Sigma notation is a way to compactly represent a sum of many similar terms, such as a Riemann sum.

Using the Sigma notation, the Riemann sum can be written $\sum_{i=1}^n f(x_i)\Delta x$.

This is read aloud as "the sum as $i = 1$ to n of f of x sub i Delta x ." The " i " is a counter, like you might have seen in a programming class.

$$\sum_{i=1}^3 a_i = a_1 + a_2 + a_3$$

Memorize

The Definite Integral:

The **definite integral** of a positive function $f(x)$ over an interval $[a, b]$ is the area between f , the x -axis, $x = a$ and $x = b$.

The **definite integral** of a positive function $f(x)$ from a to b is the area under the curve between a and b .

If $f(t)$ represents a positive rate (in y -units per t -units), then the **definite integral** of f from a to b is the total y -units that accumulate between $t = a$ and $t = b$.

Notation for the Definite Integral:

The definite integral of f from a to b is written

$$\int_a^b f(x) dx$$

The \int symbol is called an **integral sign**; it's an elongated letter S , standing for sum.

(The \int corresponds to the Σ from the Riemann sum)

The dx on the end must be included; The dx tells what the variable is – in this example, the variable is x . (The dx corresponds to the Δx from the Riemann sum)

The function f is called the **integrand**.

The a and b are called the **limits of integration**.

Verb forms:

We **integrate**, or **find the definite integral** of a function. This process is called **integration**.

Formal Algebraic Definition: $\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$. $\Delta x \rightarrow 0$

Practical Definition:

The definite integral can be approximated with a Riemann sum (dividing the area into rectangles where the height of each rectangle comes from the function, computing the area of each rectangle, and adding them up). The more rectangles you use, the narrower the rectangles are, the better your approximation will be.

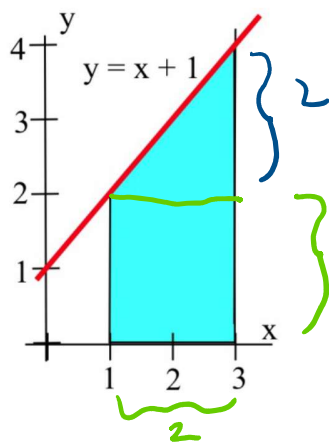
Looking Ahead:

We will have methods for computing exact values of some definite integrals from formulas soon. In many cases, including when the function is given to you as a table or graph, you will still need to approximate the definite integral with rectangles.

3.1

Example 8

Using the idea of area, determine the value of $\int_1^3 (1+x) dx$.

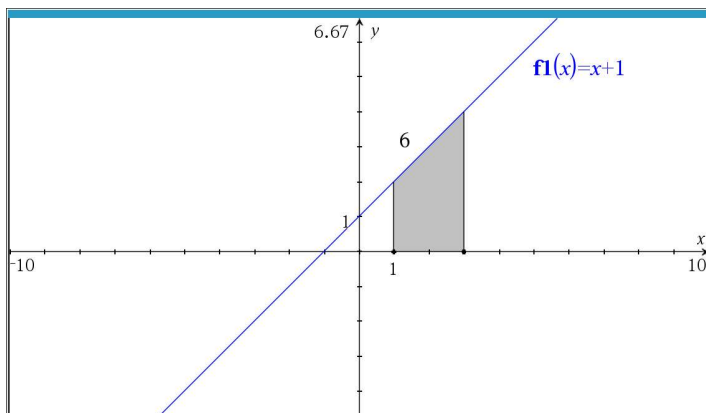


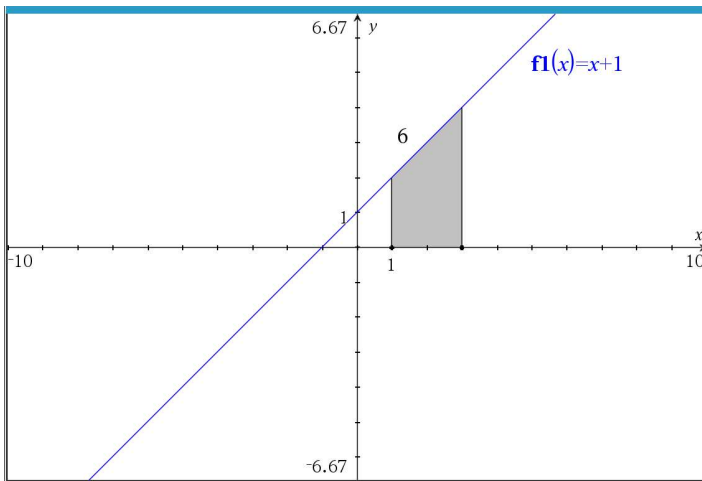
$$\text{area of } \Delta = \left(\frac{1}{2}\right)(2)(2) = 2$$

$$\text{Area of square} = 2 \times 2 = 4$$

$$\int_1^3 (x+1) dx = 2 + 4 = 6$$

2nd calc $\rightarrow \int f(x) dx \rightarrow$ lower limit, upper limit





Memorize

The Definite Integral and Signed Area:

The **definite integral** of a function $f(x)$ over an interval $[a, b]$ is the **signed area** between f , the x -axis, $x = a$ and $x = b$.

The **definite integral** of a function $f(x)$ from a to b is the **signed area** under the curve between a and b .

If the function is positive, the signed area is positive, as before (and we can call it area.)

If the function dips below the x -axis, the areas of the regions below the x -axis come in with a negative sign. In this case, we cannot call it simply “area.” These negative areas take away from the definite integral.

$$\int_a^b f(x) dx = (\text{Area above } x\text{-axis}) - (\text{Area below } x\text{-axis}).$$

If $f(t)$ represents a positive rate (in y -units per t -units), then the **definite integral** of f from a to b is the **total** y -units that accumulate between $t = a$ and $t = b$.

If $f(t)$ represents any rate (in y -units per t -units), then the **definite integral** of f from a to b is the **net** y -units that accumulate between $t = a$ and $t = b$.

Scientific Notebook

Estimate $\int_0^{10} (x^3 + 2x - 3) dx$

$x^3 + 2x - 3$ Approximate integral (left boxes) is $\sum_{i_3=0}^9 (i_3^3 + 2i_3 - 3) = 2085.0$

$x^3 + 2x - 3$ Approximate integral (right boxes) is $\sum_{i_4=1}^{10} (i_4^3 + 2i_4 - 3) = 3105.0$

$x^3 + 2x - 3$ Approximate integral (midpoint rule) is $\sum_{i_5=0}^9 \left(2i_5 + \left(i_5 + \frac{1}{2}\right)^3 - 2 \right) = 2557.5$

$\int_0^{10} (x^3 + 2x - 3) dx = 2570$ exact value

Your name MTH 261 quiz 4 write each problem.

1. In problems 1 – 10 find dy/dx by differentiating implicitly then find the value of dy/dx at the given point.

2. $x^2 + 5y^2 = 45$, point (5, 2)

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(5y^2) = \frac{d}{dx}(45)$$

$$2x + 10y \frac{dy}{dx} = 0$$

$$10y \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{10y}$$

$$\frac{dy}{dx} = -\frac{x}{5y}$$

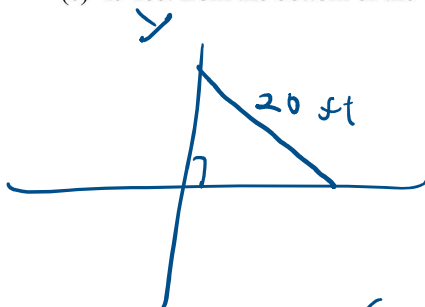
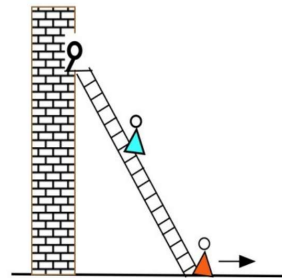
$$\frac{dy}{dx}(5, 2) = -\frac{5}{(5)(2)}$$

$$\frac{dy}{dx}(5, 2) = -\frac{1}{2}$$

#2

20. A young woman and her boyfriend plan to elope, but she must rescue him from his mother who has locked him in his room. The young woman has placed a 20 foot long ladder against his house and is knocking on his window when his mother begins pulling the bottom of the ladder away from the house at a rate of 3 feet per second. How fast is the top of the ladder (and the young couple) falling when the bottom of the ladder is

- (a) 12 feet from the bottom of the wall?
- (b) 16 feet from the bottom of the wall?
- (c) 19 feet from the bottom of the wall?



Let $x(t)$ = distance from wall to base of ladder at time t (sec)

Let $y(t)$ = height of ladder at t

Goal: find $\frac{dy}{dt}$ when $x = 12$ ft

1 Goal: find $\frac{dy}{dt}$ when $x = 12$ ft

$$\frac{dx}{dt} = \frac{3 \text{ ft}}{\text{sec}}$$

$$x^2 + y^2 = (20 \text{ ft})^2 = 400 \text{ ft}^2$$

$$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(400)$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$x \frac{dx}{dt} + y \frac{dy}{dt} = 0$$

$$y \frac{dy}{dt} = -x \frac{dx}{dt}$$

$$\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}$$

$$x^2 + y^2 = 400$$

$$x = 12 \Rightarrow y^2 = 400 - 144 = 256$$

$$\Rightarrow y = \sqrt{256}$$

$$\boxed{y = 16 \text{ ft}}$$

$$\frac{dy}{dt} = \left(-\frac{12 \text{ ft}}{16 \text{ ft}}\right) \left(\frac{3 \text{ ft}}{\text{sec}}\right)$$

$$\frac{dy}{dt} = -\frac{(3)(3)}{4} \frac{\text{ft}}{\text{sec}}$$

$$\boxed{\frac{dy}{dt} = -\frac{9}{4} \frac{\text{ft}}{\text{sec}} = -2.5 \frac{\text{ft}}{\text{sec}}}$$